Continuity and Differentiability
Application of Derivatives

HIGHLIGHTS

CONTINUITY

	Continuity	Discontinuity
At a point	• A function $f(x)$ is said to be continuous at a point $x = a$ in the domain of $f(x)$, if (i) $f(a)$ is defined (ii) $\lim_{x \to a} f(x)$ exists and (iii) $\lim_{x \to a} f(x) = f(a)$ • A function $f(x)$ is said to be continuous at a point $x = a$ in its domain, if $f(a)$ exists, $\lim_{x \to a^-} f(x)$ exists, $\lim_{x \to a^+} f(x)$ exists and $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$. The point where the function is continuous is called the point of continuity.	• If the function $f(x)$ is not continuous at $x = a$, it is said to be discontinuous at $x = a$. Here 'a' is called the point of discontinuity. • The discontinuity of a function $f(x)$ at $x = a$ may be due to any of the following reasons: (i) $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ or, both may not exist. (ii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but are not equal. (iii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist and are equal but both may not be equal to $f(a)$.
In an interval	**Continuity on an Open Interval** A function $f(x)$ is said to be continuous on an open interval (a, b), if it is continuous at each point of (a, b). **Continuity on a Closed Interval** A function $f(x)$ is said to be continuous on a closed interval $[a, b]$ if (i) $f(x)$ is continuous from right at $x = a$, *i.e.* $\lim_{h \to 0} f(a+h) = f(a)$ (ii) $f(x)$ is continuous from left at $x = b$, *i.e.* $\lim_{h \to 0} f(b-h) = f(b)$ (iii) $f(x)$ is continuous at each point of the open interval (a, b).	A real valued function $f(x)$ is said to be discontinuous if it is not continuous at atleast one point in the given interval.

Note :

- Sum, difference and product of two continuous functions are continuous.
- If $f(x)$ and $g(x)$ are two continuous functions, then $\frac{f(x)}{g(x)}$ (provided $g(x) \neq 0$), is continuous.

DIFFERENTIABILITY

Left Hand Derivative

If $y = f(x)$ is a real valued function and a is any real number, then $\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$, if it exists, is called the left hand derivative of $f(x)$ at $x = a$ and is denoted by $Lf'(a)$.

Right Hand Derivative

If $y = f(x)$ is a real valued function and a is any real number, then $\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$, if it exists, is called the right hand derivative of $f(x)$ at $x = a$ and is denoted by $Rf'(a)$.

Differentiability

A real valued function $f(x)$, is said to be differentiable at a point $x = a$ if and only if

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \text{ or } Lf'(a) = Rf'(a)$$

Note :

- Every differentiable functions is continuous but the converse is not necessarily true.
- A real valued function $f(x)$ is not differentiable
  (i) at a point $x = a$ if $Lf'(a) \neq Rf'(a)$.
  (ii) in an interval if it is not differentiable atleast one point in the interval.

SOME PROPERTIES OF DERIVATIVES

1.	Sum or Difference	$(u \pm v)' = u' \pm v'$
2.	Product Rule	$(uv)' = u'v + uv'$
3.	Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, $v \neq 0$
4.	Composite Function (Chain Rule)	(a) Let $y = f(t)$ and $t = g(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ (b) Let $y = f(t)$, $t = g(u)$ and $u = m(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}$
5.	Implicit Function	Here, we differentiate the function of type $f(x, y) = 0$.
6.	Logarithmic Function	If $y = u^v$, where u and v are the functions of x, then $\log y = v \log u$. Differentiating w.r.t. x, we get $\frac{d}{dx}(u^v) = u^v\left[\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right]$
7.	Parametric Function	If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$, $f'(t) \neq 0$
8.	Second Order Derivative	Let $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ If $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$ or $\frac{d^2y}{dx^2} = f''(x)$

SOME GENERAL DERIVATIVES

Function	Derivative	Function	Derivative	Function	Derivative
x^n	nx^{n-1}	$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\operatorname{cosec}^2 x$	$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec}x \cot x$	e^{ax}	ae^{ax}	e^x	e^x
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}};\ x\in(-1,1)$	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}};x\in(-1,1)$	$\tan^{-1} x$	$\frac{1}{1+x^2};\ x\in R$
$\cot^{-1} x$	$-\frac{1}{1+x^2};\ x\in R$	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}};\ x\in R-[-1,1]$	$\operatorname{cosec}^{-1}x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}};\ x\in R-[-1,1]$
$\log_e x$	$\frac{1}{x};\ x>0$	a^x	$a^x \log_e a;\ a>0$	$\log_a x$	$\frac{1}{x\log_e a};\ x>0$ and $a>0$

Important Theorems

Rolle's Theorem

If a real valued function $f(x)$
(i) is continuous in $[a, b]$
(ii) is differentiable on (a, b)
(iii) $f(a) = f(b)$,
then there exist at least one real number c in the interval (a, b) such that $f'(c) = 0$.

Lagrange's Mean Value Theorem

If a real valued function $f(x)$ is
(i) continuous in $[a, b]$
(ii) differentiable on (a, b),
then there exists at least one real number c, where $a<c<b$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

APPLICATION OF DERIVATIVES

RATE OF CHANGE OF QUANTITIES

Let $y = f(x)$ be a function. Then $\frac{dy}{dx}$ denotes the rate of change of y w.r.t. x.

INCREASING AND DECREASING FUNCTIONS

A function $f(x)$ defined on (a, b) is said to be

		Using derivative test
Increasing Function	If $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ or $x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2)$ for all $x_1, x_2 \in (a, b)$.	If $f'(x) \ge 0$ for each $x \in (a, b)$.
Strictly Increasing Function	If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ or $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$.	If $f'(x) > 0$ for each $x \in (a, b)$.
Decreasing Function	If $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$ or $x_1 > x_2 \Rightarrow f(x_1) \le f(x_2)$ for all $x_1, x_2 \in (a, b)$.	If $f'(x) \le 0$ for each $x \in (a, b)$.
Strictly Decreasing Function	If $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ or $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$.	If $f'(x) < 0$ for each $x \in (a, b)$.

TANGENTS AND NORMALS

Let $y = f(x)$ be a curve. Then, slope and equation of tangent and normal at (x_1, y_1) is given by,

	Tangent	Normal
Slope	$\left.\frac{dy}{dx}\right\|_{(x_1,y_1)}$	$\frac{-1}{\left.\left(\frac{dy}{dx}\right)\right\|_{(x_1,y_1)}}$
Equations	$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1)$	$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}(x - x_1)$

Remarks

- If $\frac{dy}{dx} = 0$ at (x_1, y_1), then tangent is parallel to x-axis. Hence, equation of tangent is $y = y_1$.
- If $\frac{dy}{dx} = \infty$ at (x_1, y_1), then tangent is perpendicular to x-axis. Hence, equation of tangent is $x = x_1$.
- Angle between two curves is $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$, where m_1 and m_2 are slopes of tangents at the point of intersection of curve.

APPROXIMATIONS

Let $y = f(x)$ be a differentiable function. Δx and Δy be the small changes in x and y respectively. Then, we find the approximate value of certain quantity by the following steps :

I. Find Δx and x

II. $\Delta y = \frac{dy}{dx}(\Delta x)$

III. $\Delta y = f(x + \Delta x) - f(x)$ or $f(x + \Delta x) = f(x) + \Delta y$

MAXIMA AND MINIMA

Maximum value of $f(x)$: Let $f(x)$ be a function defined on an interval I. Then f is said to have maximum value in I, if there exists a point $a \in I$, such that $f(a) \geq f(x)$ $\forall\ x \in I$.

Minimum value of $f(x)$: Let $f(x)$ be a function defined on an interval I. Then f is said to have minimum value in I, if there exists a point $a \in I$, such that $f(a) \leq f(x)$ $\forall\ x \in I$.

LOCAL MAXIMA AND LOCAL MINIMA

To find the local maximum or local minimum values of a function, the following tests are useful.

I. First derivative test : A function $f(x)$ is said to have a local maximum value at $x = a$, if

(i) $f'(a) = 0$

(ii) $f'(a - h) > 0$

(iii) $f'(a + h) < 0$, where h is small positive number.

A function $f(x)$ is said to have a local minimum value at $x = a$ if

(i) $f'(a) = 0$

(ii) $f'(a - h) < 0$

(iii) $f'(a + h) > 0$, where h is small positive number.

Note : $f(a)$ is neither local maximum nor minimum, if $f'(a) = 0$ and $f'(a - h) > 0$, $f'(a + h) > 0$ or $f'(a - h) < 0$, $f'(a + h) < 0$. In this case, $x = a$ is called the point of inflection.

II. Second derivative test : A function $y = f(x)$ is said to have a local maximum value at $x = a$ if $f'(a) = 0$ and $f''(a) < 0$ and a local minimum value at $x = a$ if $f'(a) = 0$ and $f''(a) > 0$.

Note : If $f''(a) = 0$, then second derivative test fails.

PROBLEMS

Very Short Answer Type

1. Show that $f(x) = x^3$ is continuous at $x = 2$.
2. If $y = x^4 + 10$ and x changes from 2 to 1.99, find the approximate change in y.
3. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing on $(-1, 1)$.
4. The total revenue received from the sale of x units of a product is given by $R(x) = 3x^2 + 40x + 10$. Find the marginal revenue when $x = 5$.
5. Find the derivative of $(4x^3 - 5x^2 + 1)^4$ w.r.t. to x.

Long Answer Type-I

6. Verify Rolle's Theorem for the function $f(x) = (x - 1)(x - 2)^2$ in [1,2].

7. Find the intervals in which the function $f(x) = \dfrac{4x^2+1}{x} (x \neq 0)$ is
(i) increasing (ii) decreasing

8. For what choices of a and b, the function $f(x) = \begin{cases} x^2, & x \le c \\ ax+b, & x > c \end{cases}$ is differentiable at $x = c$.

9. Differentiate $(e^x \cos^3 x \sin^2 x)$ w.r.t. x.

10. An open box is to be made out of a piece of cardboard measuring (24 cm × 24 cm) by cutting off equal squares from the corners and turning up the sides. Find the height of the box when it has maximum volume.

Long Answer Type-II

11. Find the equations of the tangent and normal to the curve $16x^2 + 9y^2 = 144$ at (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$. Also, find the point of intersection where both tangent and normal cut the x-axis.

12. If $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}, x \neq \dfrac{\pi}{4}$, find the value of $f\left(\dfrac{\pi}{4}\right)$ so that $f(x)$ becomes continuous at $x = \dfrac{\pi}{4}$.

13. Prove that the height and the radius of the base of an open cylinder of given surface area and maximum volume are equal.

14. If $(x - a)^2 + (y - b)^2 = c^2$, for some $c > 0$, prove that
$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$
is a constant independent of a and b.

15. Verify Lagrange's mean value theorem for the function $f(x) = x(x - 1)(x - 2)$ in the interval $\left[0, \dfrac{1}{2}\right]$.

SOLUTIONS

1. We have, $f(2) = 2^3 = 8$;

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0}(2+h)^3 = \lim_{h\to 0}(8+h^3+12h+6h^2) = 8;$$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0}(2-h)^3 = \lim_{h\to 0}(8-h^3-12h+6h^2) = 8$$

$$\because \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$$

Hence, $f(x)$ is continuous at $x = 2$.

2. Given $y = x^4 + 10$ ...(i)

$$\therefore \frac{dy}{dx} = 4x^3 \quad \text{...(ii)}$$

Approximate change in y is given by

$$\Delta y = \frac{dy}{dx}\Delta x = 4x^2 \Delta x \quad \text{... (iii)}$$

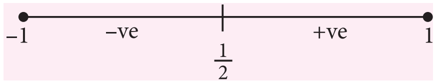
Putting $x = 2$ and $x + \Delta x = 1.99 \Rightarrow \Delta x = -0.01$ in (iii), we get

$\Delta y = 4(2)^2 \cdot (-0.01) = -0.16$

3. Given, $f(x) = x^2 - x + 1$

$\therefore \quad f'(x) = 2x - 1$

Sign scheme for $f'(x)$ in $(-1, 1)$ is



Since $f'(x)$ changes sign in $(-1, 1)$, therefore, it is neither increasing nor decreasing in $(-1, 1)$.

4. We have, $R(x) = 3x^2 + 40x + 10$

$$\Rightarrow \text{MR} = \frac{dR}{dx} = \frac{d}{dx}(3x^2+40x+10) = 6x+40$$

$\Rightarrow \quad [\text{MR}]_{x=5} = (6 \times 5 + 40) = 70$

Hence, the required marginal revenue is ₹70.

5. $y = u^4$, where $u = 4x^3 - 5x^2 + 1$

Now, $\dfrac{dy}{du} = 4u^3$ and $\dfrac{du}{dx} = 12x^2 - 10x$

By chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$$\therefore \frac{dy}{dx} = 4u^3(12x^2-10x)$$
$$= 4(4x^3 - 5x^2 + 1)^3(12x^2 - 10x)$$

6. The given function is

$f(x) = (x - 1)(x - 2)^2, x \in [1,2]$

Clearly $f(x)$ is a polynomial in x, therefore, $f(x)$ is continuous and differentiable everywhere.

$\therefore \quad f(x)$ is continuous on [1, 2]

and $f(x)$ is differentiable on (1, 2)

Also $f(1) = 0$ and $f(2) = 0$

Hence all conditions of Rolle's theorem are satisfied for $f(x)$ in [1, 2]

$\Rightarrow \quad \exists\, c \in (1, 2)$ satisfying $f'(c) = 0$

Now $f'(x) = 1\cdot(x - 2)^2 + (x - 1) \cdot 2(x - 2)$

$= (x - 2)(x - 2 + 2x - 2)$

$= (x - 2)(3x - 4)$

$\because \quad f'(c)=0 \Rightarrow c=2 \text{ or } c=\frac{4}{3}$

But $c = 2 \notin]1, 2[$, therefore, $c=\frac{4}{3}\in]1,2[$ satisfying $f'(c) = 0$

Thus, Rolle's theorem has been verified.

7. We have, $f(x)=\frac{4x^2+1}{x}, (x \neq 0)$

Differentiating w.r.t. to x, we get

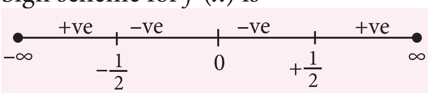
$$f'(x)=4-\frac{1}{x^2}=\frac{4x^2-1}{x^2}$$

For $f'(x) = 0 \Rightarrow 4x^2 - 1 = 0$

$\Rightarrow \quad (2x - 1)(2x + 1) = 0$

$\Rightarrow \quad x=\frac{1}{2} \text{ or } x=\frac{-1}{2}$

Sign scheme for $f'(x)$ is



(i) Now for $f(x)$ to be increasing, $f'(x) \geq 0$

$\therefore \quad f(x)$ is increasing in $\left(-\infty,-\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$

Since, $f(x)$ is continuous at $x=\frac{-1}{2}$ and $x=\frac{1}{2}$.

$\therefore \quad f(x)$ is increasing in $\left(-\infty,\frac{-1}{2}\right]\cup\left[\frac{1}{2},\infty\right)$

(ii) For $f(x)$ to be decreasing, $f'(x) \leq 0$

$\therefore \quad f(x)$ is decreasing in $\left(-\frac{1}{2},0\right)\cup\left(0,\frac{1}{2}\right)$

Since, $f(x)$ is continuous at $x=-\frac{1}{2}$ and $x=\frac{1}{2}$

$\therefore \quad f(x)$ is decreasing in $\left[-\frac{1}{2},0\right)\cup\left(0,\frac{1}{2}\right]$

Here, 0 is excluded as $f(x)$ is not defined at $x = 0$

8. It is given that $f(x)$ is differentiable at $x = c$ and every differentiable function is continuous. So, $f(x)$ is continuous at $x = c$.

$\therefore \quad \lim_{x\to c^-} f(x)=\lim_{x\to c^+} f(x)=f(c)$

$\Rightarrow \quad \lim_{x\to c^-} (x^2)=\lim_{x\to c^+} (ax+b)=c^2$

$\Rightarrow \quad c^2 = ac + b$ ... (i)

Now, $f(x)$ is differentiable at $x = c$

$\therefore \quad Lf'(c) = Rf'(c)$

$\Rightarrow \quad \lim_{x\to c^-} =\frac{f(x)-f(c)}{x-c}=\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}$

$\Rightarrow \quad \lim_{x\to c^-}\frac{x^2-c^2}{x-c}=\lim_{x\to c^+}\frac{(ax+b)-c^2}{x-c}$

$\Rightarrow \quad \lim_{x\to c}(x+c)=\lim_{x\to c}\frac{ax+b-(ac+b)}{x-c}$ [From (i)]

$\Rightarrow \quad \lim_{x\to c}(x+c)=\lim_{x\to c}\frac{a(x-c)}{x-c}$

$\Rightarrow \quad \lim_{x\to c}(x+c)=\lim_{x\to c}(a) \Rightarrow 2c = a$ ... (ii)

From (i) and (ii) we get

$a = 2c$ and $b = -c^2$

9. Let $y = e^x\cos^3 x\sin^2 x$

Taking log on both sides, we get

$\log y = x + 3 \log \cos x + 2\log \sin x$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=1+\frac{3}{\cos x}\cdot(-\sin x)+\frac{2}{\sin x}\cdot\cos x$$

$$\Rightarrow \quad \frac{dy}{dx}=y.\{1-3\tan x+2\cot x\}$$

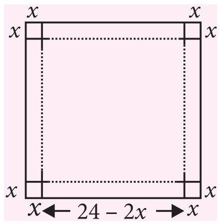
$$= (e^x\cos^3 x\sin^2 x)(1 - 3\tan x + 2\cot x).$$

10. Let the length of the side of the each square cut off from the corners be x cm.

Then, height of the box = x cm.

Volume $V = (24 - 2x)^2 \times x$

$= 4x^3 - 96x^2 + 576x$



$\Rightarrow \quad \frac{dV}{dx}=12(x^2-16x+48)$

and $\frac{d^2V}{dx^2}=24(x-8)$

Now, for maximum or minimum volume,

$\frac{dV}{dx}=0 \Rightarrow x^2-16x+48=0$

$\Rightarrow \quad (x - 12)(x - 4) = 0 \Rightarrow x = 4 \; [\because x \neq 12]$

$\left[\frac{d^2V}{dx^2}\right]_{x=4}=-96<0$

$\therefore \quad V$ is maximum at $x = 4$.

So, required height = 4 cm.

11. Given curve is $16x^2 + 9y^2 = 144$ ... (i)

Differentiating w.r.t. x, we get

$$32x+18y\frac{dy}{dx}=0$$

$\Rightarrow \quad \frac{dy}{dx}=-\frac{16}{9}\cdot\frac{x}{y}$ ... (ii)

Now, $P(2, y_1)$ lies on (i), therefore,

$$16 \times 4 + 9y_1^2 = 144$$

$$\Rightarrow \quad 9y_1^2 = 80 \Rightarrow y_1 = \frac{4\sqrt{5}}{3} \qquad [\because y_1 > 0]$$

$$\therefore \quad P \equiv \left(2, \frac{4\sqrt{5}}{3}\right)$$

At point P, $\dfrac{dy}{dx} = -\dfrac{16}{9} \times \dfrac{2}{\frac{4\sqrt{5}}{3}} = -\dfrac{8}{3\sqrt{5}}$

$\therefore$ Equation of the tangent at P is

$$y - \frac{4\sqrt{5}}{3} = -\frac{8}{3\sqrt{5}}(x-2)$$

This meets x-axis where $y = 0$

$$\therefore \quad 0 - \frac{4\sqrt{5}}{3} = -\frac{8}{3\sqrt{5}}(x-2)$$

$$\Rightarrow \quad \frac{5}{2} = x - 2 \text{ or } x = \frac{9}{2}$$

$\therefore$ Tangent meets x-axis at $\left(\frac{9}{2}, 0\right)$

Also equation of normal at P is

$$y - \frac{4\sqrt{5}}{3} = \frac{3\sqrt{5}}{8}(x-2)$$

This meet x-axis where $y = 0$

$$\therefore \quad 0 - \frac{4\sqrt{5}}{3} = \frac{3\sqrt{5}}{8}(x-2)$$

$$\text{or} \quad -\frac{32}{9} = x - 2 \text{ or } x = -\frac{14}{9}$$

$\therefore$ Normal meets x-axis at $\left(-\frac{14}{9}, 0\right)$.

12. Given, $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}, x \neq \dfrac{\pi}{4}$

$$\therefore \quad \lim_{x \to \frac{\pi}{4}} f(x) = \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$$

Putting $x = \dfrac{\pi}{4} + h$ so that as $x \to \dfrac{\pi}{4}, h \to 0$

$$= \lim_{h \to 0} \frac{\sqrt{2}\cos\left(\frac{\pi}{4} + h\right) - 1}{\cot\left(\frac{\pi}{4} + h\right) - 1}$$

$$= \lim_{h \to 0} \frac{\sqrt{2}\left(\cos\frac{\pi}{4}\cos h - \sin\frac{\pi}{4}\sin h\right) - 1}{\frac{1 - \tan h}{1 + \tan h} - 1}$$

$$= \lim_{h \to 0} \frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos h - \frac{1}{\sqrt{2}}\sin h\right) - 1}{-2\tan h}(1 + \tan h)$$

$$= \lim_{h \to 0} \frac{\cos h - \sin h - 1}{-2\tan h}(1 + \tan h)$$

$$= \lim_{h \to 0} \frac{1 - \cos h + \sin h}{2\tan h}(1 + \tan h)$$

$$= \lim_{h \to 0} \frac{2\sin^2\frac{h}{2} + 2\sin\frac{h}{2}\cos\frac{h}{2}}{2\tan h}(1 + \tan h)$$

$$= \lim_{h \to 0} \frac{\sin\frac{h}{2}\left(\sin\frac{h}{2} + \cos\frac{h}{2}\right)}{\tan h}(1 + \tan h)$$

$$= \lim_{h \to 0} \frac{\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\frac{h}{2}\left(\sin\frac{h}{2} + \cos\frac{h}{2}\right)}{\left(\frac{\tan h}{h}\right) \cdot h}(1 + \tan h)$$

$$= \frac{1}{2}(0+1)(1+0) = \frac{1}{2} \quad \therefore \quad \lim_{x \to \frac{\pi}{4}} f(x) = \frac{1}{2}$$

Hence, for $f(x)$ to be continuous at $x = \dfrac{\pi}{4}, f\left(\dfrac{\pi}{4}\right) = \dfrac{1}{2}$

13. Let x be the radius of the cylinder and z be its height, let S be the area of the surface of the cylinder and V be its volume.

Given S = constant

Now, $S = 2\pi xz + \pi x^2$ ... (i)

and $V = \pi x^2 z$ ...(ii)

From (i), $2\pi xz = S - \pi x^2$

$$\therefore \quad z = \frac{S - \pi x^2}{2\pi x} \qquad ...(iii)$$

From (ii), $V = \pi x^2\left(\dfrac{S - \pi x^2}{2\pi x}\right) = \dfrac{Sx}{2} - \dfrac{\pi x^3}{2}$

$$\therefore \quad \frac{dV}{dx} = \frac{S}{2} - \frac{3\pi x^2}{2}$$

For maximum or minimum values of V,

$$\frac{dV}{dx} = 0 \Rightarrow \frac{S}{2} - \frac{3\pi}{2}x^2 = 0$$

$$\Rightarrow \quad x^2 = \frac{S}{3\pi} \quad \Rightarrow \quad x = \sqrt{\frac{S}{3\pi}}$$

Now, $\frac{d^2V}{dx^2}=-\frac{3\pi}{2}\cdot 2x=-3\pi x$

When $x=\sqrt{\frac{S}{3\pi}}, \frac{d^2V}{dx^2}=-3\pi\sqrt{\frac{S}{3\pi}}=-\sqrt{3\pi S}<0$

Hence when $x=\sqrt{\frac{S}{3\pi}}$, V is maximum.

Now, $\frac{x}{z}=\frac{2\pi x^2}{5-\pi x^2} \Rightarrow \frac{x}{z}=\frac{2\pi\cdot\frac{S}{3\pi}}{S-\pi\frac{S}{3\pi}}=\frac{\frac{2}{3}S}{\frac{2}{3}S}=1 \Rightarrow x=z$

Thus for maximum volume of a cylinder to given surface area $x = z$.

14. Given, $(x - a)^2 + (y - b)^2 = c^2$ ...(i)

Differentiating both sides w.r.t. x, we get

$$2(x-a)+2(y-b)\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x-a}{y-b} \quad \text{... (ii)}$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=-\frac{(y-b).1-(x-a)\frac{dy}{dx}}{(y-b)^2}$$

$$=-\frac{(y-b)-(x-a)\left\{-\left(\frac{x-a}{y-b}\right)\right\}}{(y-b)^2} \quad \text{[From (ii)]}$$

$$=-\frac{\frac{(x-a)^2}{y-b}+(y-b)}{(y-b)^2}=-\frac{(x-a)^2+(y-b)^2}{(y-b)^3}$$

$$=-\frac{c^2}{(y-b)^3} \quad \text{...(iii)}$$

Now, $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}=\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{3/2}$

$$=\left[\frac{(x-a)^2+(y-b)^2}{(y-b)^2}\right]^{3/2} \frac{(c^2)^{3/2}}{(y-b)^3}=\frac{c^3}{(y-b)^3} \quad \text{...(iv)}$$

Now from (iii) and (iv), we have

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=-c$$

which is a constant independent of a and b.

15. Given $f(x) = x(x - 1)(x - 2) = x(x^2 - 3x + 2)$

$\therefore$ $f'(x) = 1\cdot(x^2 - 3x + 2) + x(2x - 3)$
$= 3x^2 - 6x + 2$

Clearly $f(x)$ is a polynomial in x, therefore, it is continuous and differentiable for all x

Hence, $f(x)$ is continuous in $\left[0,\frac{1}{2}\right]$

and $f(x)$ is differentiable in $\left(0,\frac{1}{2}\right)$

Now $f(0) = 0$ and $f\left(\frac{1}{2}\right)=\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)=\frac{3}{8}$

Hence, all conditions of Lagrange's mean value theorem are satisfied for $f(x)$ in $\left[0,\frac{1}{2}\right]$.

$\therefore$ There exists at least one $c\in\left[0,\frac{1}{2}\right]$ such that

$$f'(c)=\frac{f\left(\frac{1}{2}\right)-f(0)}{\frac{1}{2}-0}$$

$$\Rightarrow 3c^2-6c+2=\frac{\frac{3}{8}-0}{\frac{1}{2}}=\frac{3}{4} \Rightarrow 12c^2 - 24c + 5 = 0$$

$$\therefore \quad c=\frac{24\pm\sqrt{576-240}}{24}=\frac{24\pm\sqrt{336}}{24}$$

$$=\frac{24\pm4\sqrt{21}}{24}=1\pm\frac{\sqrt{21}}{6}$$

Hence, $c=1+\frac{\sqrt{21}}{6}, 1-\frac{\sqrt{21}}{6}$

But $0<c<\frac{1}{2}$ $\therefore$ $c=1-\frac{\sqrt{21}}{6}$

Thus Lagrange's mean value theorem has been verified.