CONTINUITY

A function $f(x)$ will be continuous at a point $x = a$, if there is no break or cut or hole or gap in the graph of the function $y = f(x)$ at the point $(a, f(a))$. Otherwise, it is discontinuous at that point.

A function f is said to be continuous at the point $x = a$ if the following conditions are satisfied :

(i) $f(a)$ exists.

(ii) $\lim_{x \to a} f(x)$ exists.

(iii) $\lim_{x \to a} f(x) = f(a)$.

CONTINUITY OF A FUNCTION ON AN INTERVAL

Continuity on an Open Interval

A function $f(x)$ is said to be continuous on an open interval (a, b), if it is continuous at each point of (a, b).

Continuity on a Closed Interval

A function $f(x)$ is said to be continuous on a closed interval $[a, b]$ if

(i) $f(x)$ is continuous from right at $x = a$, *i.e.* $\lim_{h \to 0} f(a + h) = f(a)$

(ii) $f(x)$ is continuous from left at $x = b$, *i.e.* $\lim_{h \to 0} f(b - h) = f(b)$

(iii) $f(x)$ is continuous at each point of the open interval (a, b).

DISCONTINUITY OF A FUNCTION

1.	At a point	A real valued function $f(x)$ is said to be discontinuous at $x = a$, if it is not continuous at $x = a$. The discontinuity may be due to any of the following reasons: (i) $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ or both may not exist. (ii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but are not equal. (iii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are equal but both may not be equal to $f(a)$.
2.	In an interval	A real valued function $f(x)$ is said to be discontinuous if it is not continuous at atleast one point in the given interval.

TYPES OF DISCONTINUITY

1.	Removable discontinuity	$f(x)$ is said to have a removable discontinuity at $x = a$ if $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a^+} f(x) \neq f(a)$ or if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$
2.	Non-removable discontinuity	$f(x)$ is said to have non-removabole discontinuity at $x = a$; (i) First kind: If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ (ii) Second kind: If $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ or both do not exist.

ALGEBRA OF CONTINUOUS FUNCTIONS

Let $f(x)$ and $g(x)$ be two continuous functions on their common domain D and let c be a real number. Then

(i) $f + g$ is continuous at $x = c$

(ii) $f - g$ is continuous at $x = c$

(iii) fg is continuous at $x = c$

(iv) $\frac{f}{g}$ is continuous at $x = c$

Note :

- If f and g are real functions such that fog is defined and if g is continuous at a point a and f is continuous at $g(a)$, then fog is continuous at $x = a$.

DIFFERENTIABILITY

Let $f(x)$ be a real function and a be any real number. Then, we define

(i) **Right-hand derivative :** $\lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$, if it exists, is called the right-hand derivative of $f(x)$ at $x = a$, and is denoted by $Rf'(a)$.

(ii) **Left-hand derivative :** $\lim_{h \to 0^-} \frac{f(a-h)-f(a)}{-h}$, if it exists, is called the left-hand derivative of $f(x)$ at $x = a$, and is denoted by $Lf'(a)$.

A function $f(x)$ is said to be differentiable at $x = a$, if $Rf'(a) = Lf'(a)$.

The common value of $Rf'(a)$ and $Lf'(a)$ is denoted by $f'(a)$ and it is known as the derivative of $f(x)$ at $x = a$.

If, however, $Rf'(a) \neq Lf'(a)$ we say that $f(x)$ is not differentiable at $x = a$.

Note :

- $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point.
- If a function is differentiable at a point, then it is necessarily continuous at that point. But the converse is not necessarily true.
- A function f is said to be a differentiable function if it is differentiable at every point in its domain.

DERIVATIVE OF A FUNCTION

If a function $f(x)$ is differentiable at every point in its domain, then

$$\lim_{h \to 0^+} \frac{f(x+h)-f(x)}{h} \text{ or } \lim_{h \to 0^-} \frac{f(x-h)-f(x)}{-h}$$ is called the derivative or differentiation of f at x and is denoted by $f'(x)$ or $\frac{d}{dx}f(x)$.

SOME PROPERTIES OF DERIVATIVE

1.	Sum or Difference	$(u \pm v)' = u' \pm v'$
2.	Product Rule	$(uv)' = u'v + uv'$
3.	Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}, v \neq 0$
4.	Composite Function (Chain Rule)	(a) Let $y = f(t)$ and $t = g(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ (b) Let $y = f(t)$, $t = g(u)$ and $u = m(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}$
5.	Parametric Function	If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}, f'(t) \neq 0$
6.	Second Order Derivative	Let $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ If $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$ or $\frac{d^2y}{dx^2} = f''(x)$
7.	Logarithmic Function	If $y = u^v$, where u and v are the functions of x, then $\log y = v \log u$. Differentiating w.r.t. x, we get $\frac{d}{dx}(u^v) = u^v\left[\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right]$
8.	Implicit Function	Here, we differentiate the function of type $f(x, y) = 0$.

SOME GENERAL DERIVATIVES

Function	Derivative	Function	Derivative	Function	Derivative
x^n	nx^{n-1}	$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\text{cosec}^2 x$	$\sec x$	$\sec x \tan x$
$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$	e^{ax}	ae^{ax}	e^x	e^x
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}};\ x\in(-1,1)$	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}};x\in(-1,1)$	$\tan^{-1} x$	$\frac{1}{1+x^2};\ x\in R$
$\cot^{-1} x$	$-\frac{1}{1+x^2};\ x\in R$	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}};\ x\in R-[-1,1]$	$\text{cosec}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}};\ x\in R-[-1,1]$
$\log_e x$	$\frac{1}{x};\ x>0$	a^x	$a^x \log_e a;\ a>0$	$\log_a x$	$\frac{1}{x\log_e a};\ x>0$ and $a>0,$

SOME IMPORTANT THEOREMS

Rolle's theorem

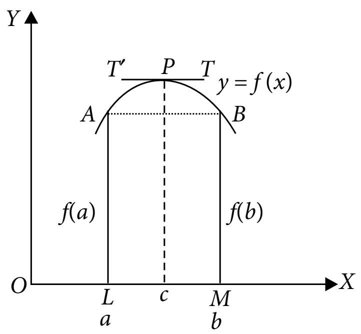
If a function $f(x)$ is

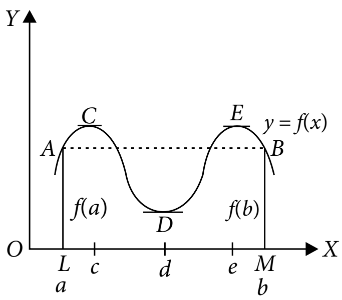
(i) continuous in the closed interval $[a, b]$ *i.e.* continuous at each point in the interval $[a, b]$

(ii) differentiable in an open interval (a, b) *i.e.* differentiable at each point in the open interval (a, b) and

(iii) $f(a) = f(b)$, then there will be at least one point c, in the interval (a, b) such that $f'(c) = 0$.

Geometrical meaning of Rolle's theorem





If the graph of a function $y = f(x)$ be continuous at each point from the point $A(a, f(a))$ to the point $B(b, f(b))$ and tangent to the graph at each point between A and B is unique *i.e.* tangent at each point between A and B exists and ordinates *i.e.* y co-ordinates of points A and B are equal, then there will be at least one point P on the curve between A and B at which tangent will be parallel to x-axis.

Lagrange's mean value theorem

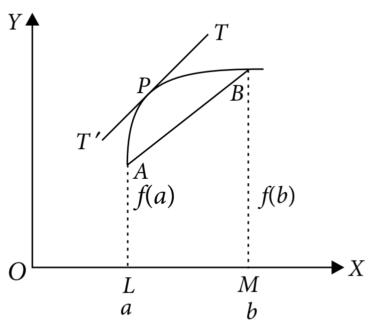
If a function $f(x)$ is

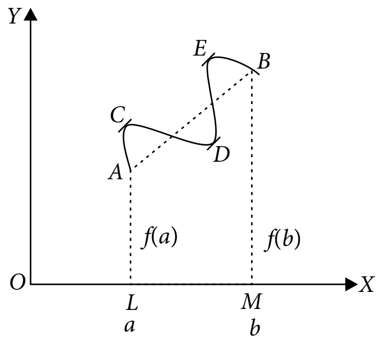
(i) continuous in the closed interval $[a, b]$ *i.e.* continuous at each point in the interval $[a, b]$

(ii) differentiable in the open interval (a, b) *i.e.* differentiable at each point in the interval (a, b) then there will be at least one point c, where

$$a < c < b \text{ such that } f'(c) = \frac{f(b)-f(a)}{b-a}$$

Geometrical meaning of Lagrange's mean value theorem





If the graph of a function $y = f(x)$ be continuous at each point from the point $A\ (a, f(a))$ to the point $B\ (b, f(b))$ and tangent at each point between A and B exists *i.e.* tangent is unique then there will be at least one point P on the curve between A and B, where tangent will be parallel to chord AB.

Very Short Answer Type

1. Discuss the continuity of the function $f(x) = \sin x - \cos x$
2. Differentiate $\cos(\sin x)$ with respect to x.
3. If $xy = x^3 + y^3$, find $\dfrac{dy}{dx}$.
4. Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.
5. Is the function defined by $f(x) = \begin{cases} x, \text{if } x \le 1 \\ 5, \text{ if } x > 1 \end{cases}$ continuous at $x = 1$?

Short Answer Type

6. If the function $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, \text{ if } x \neq \dfrac{\pi}{2} \\ 3, \quad \text{if } x = \dfrac{\pi}{2} \end{cases}$

   is continuous at $x = \dfrac{\pi}{2}$, then find the value of k.
7. If $y^x = e^{y-x}$, then find the value of $\dfrac{dy}{dx}$.
8. Show that $f(x) = [x]$ is not differentiable at $x = 1$.
9. If $y = \sqrt{e^{\sqrt{x}}}$, find $\dfrac{dy}{dx}$.
10. If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$, prove that
$$\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$$

Long Answer Type

11. If $y = e^x \sin x^3 + (\tan x)^x$, find $\dfrac{dy}{dx}$.
12. If $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, find $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{3}$.
13. Verify Lagrange's mean value theorem for the function $f(x) = x(x-1)(x-2)$ in the interval $\left[0, \dfrac{1}{2}\right]$.
14. (i) If $y = b\tan^{-1}\left(\dfrac{x}{a} + \tan^{-1}\dfrac{y}{x}\right)$, find $\dfrac{dy}{dx}$.

    (ii) If $\sqrt{x} + \sqrt{y} = 4$, find $\left.\dfrac{dy}{dx}\right|_{\text{at } x=1}$.
15. (i) If Rolle's theorem hold for the function $f(x) = x^3 + bx^2 + ax + 5$ on $[1, 3]$ where $c = \left(2 + \dfrac{1}{\sqrt{3}}\right)$, find the values of a and b.

    (ii) Using Rolle's theorem, find at what points on the curve $y = x^2$ on $[-2, 2]$ is the tangent parallel to x-axis.

SOLUTIONS

1. Since $\sin x$ and $\cos x$ are continuous functions and difference of two continuous functions is a continuous function, therefore $\sin x - \cos x$ *i.e.*, $f(x)$ is a continuous function.

2. Let $y = \cos(\sin x)$

   Now, $\dfrac{dy}{dx} = \dfrac{d\{\cos(\sin x)\}}{dx}$
   $= -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$

3. Given, $xy = x^3 + y^3$

   Differentiating w.r.t. x, we get
$$\frac{d}{dx}(xy) = \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3)$$
   or $1 \cdot y + x \cdot \dfrac{dy}{dx} = 3x^2 + 3y^2\dfrac{dy}{dx}$

   $\Rightarrow (x - 3y^2)\dfrac{dy}{dx} = 3x^2 - y \Rightarrow \dfrac{dy}{dx} = \dfrac{3x^2 - y}{x - 3y^2}$

4. $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3}(2x^2 - 1) = 17$

   $f(3) = 17$

   $\therefore$ f is continuous at $x = 3$.

5. At $x = 1$,

$$\lim_{x\to1^-} f(x) = \lim_{x\to1} x = 1, \quad \lim_{x\to1^+} f(x) = \lim_{x\to1} 5 = 5$$

$$\lim_{x\to1^-} f(x) \neq \lim_{x\to1^+} f(x)$$

$\therefore$ f is discontinuous at $x = 1$.

6. Here, $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

$$\therefore \text{ L.H.L.} = \lim_{x\to\left(\frac{\pi}{2}\right)^-} f(x) = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{k\cos x}{\pi - 2x}$$

$$= \lim_{h\to0} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h\to0} \frac{k\sin h}{2h}$$

$$= \lim_{h\to0} \frac{k}{2} \times \frac{\sin h}{h} = \frac{k}{2} \times 1 = \frac{k}{2} \quad \left(\because \lim_{x\to0} \frac{\sin x}{x} = 1\right)$$

$$\text{R.H.L.} = \lim_{x\to\left(\frac{\pi}{2}\right)^+} f(x) = \lim_{x\to\left(\frac{\pi}{2}\right)^+} \frac{k\cos x}{\pi - 2x}$$

$$= \lim_{h\to0} \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = \lim_{h\to0} \frac{-k\sin h}{-2h}$$

$$= \lim_{h\to0} \frac{k}{2} \times \frac{\sin h}{h} = \frac{k}{2} \times 1 = \frac{k}{2} \quad \left(\because \lim_{x\to0} \frac{\sin x}{x} = 1\right)$$

Also, $f\left(\frac{\pi}{2}\right) = 3.$

Since, $f(x)$ is continuous at $x = \frac{\pi}{2}$

$$\therefore \text{ L.H.L.} = \text{R.H.L.} = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{k}{2} = 3 \Rightarrow k = 6$$

7. Here, $y^x = e^{y - x}$

Taking log on both sides, we get

$\log y^x = \log e^{y - x}$

$\Rightarrow$ $x \log y = (y - x) \log e \Rightarrow x \log y = y - x$ ...(i)

On differentiating w.r.t. x, we get

$$\left[x\frac{d}{dx}(\log y) + \log y\frac{d}{dx}(x)\right] = \frac{dy}{dx} - 1$$

(Using product rule)

$$\Rightarrow x\left(\frac{1}{y}\right)\frac{dy}{dx} + \log y(1) = \frac{dy}{dx} - 1$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{x}{y} - 1\right) = -1 - \log y$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{y}{(1 + \log y)y} - 1\right] = -(1 + \log y)$$

$$\Rightarrow \left[\because \text{ from (i), } x = \frac{y}{(1 + \log y)}\right]$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{1 - 1 - \log y}{1 + \log y}\right] = -(1 + \log y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1 + \log y)^2}{\log y}$$

8. We have, $Rf'(1) = \lim_{h\to0^+} \dfrac{f(1 + h) - f(1)}{h}$

$$= \lim_{h\to0^+} \frac{[1 + h] - [1]}{h} = 0 \quad (\because [1 + h] = 1 \text{ and } [1] = 1)$$

and $Lf'(1) = \lim_{h\to0^-} \dfrac{f(1 - h) - f(1)}{-h}$

$$= \lim_{h\to0^-} \frac{[1 - h] - [1]}{-h} = \infty$$

$\{\because [1 - h] = 0$ and $[1] = 1\}$.

Thus $Rf'(1) \neq Lf'(1)$.

Hence, $f(x) = [x]$ is not differentiable at $x = 1$.

9. Putting $\sqrt{x} = t, e^{\sqrt{x}} = e^t = u$ ...(i)

we get, $y = \sqrt{e^{\sqrt{x}}} = \sqrt{u}$

$$\Rightarrow \frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}, \quad \because u = e^t \Rightarrow \frac{du}{dt} = e^t$$

and $t = \sqrt{x} \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \left(\frac{1}{2\sqrt{u}} \times e^t \times \frac{1}{2\sqrt{x}}\right) = \left\{\frac{1}{2\sqrt{u}} \times u \times \frac{1}{2\sqrt{x}}\right\}$$

$$= \frac{\sqrt{u}}{4\sqrt{x}} = \frac{e^{\frac{1}{2}t}}{4\sqrt{x}} = \frac{e^{\frac{1}{2}\sqrt{x}}}{4\sqrt{x}}. \quad \text{(Using (i))}$$

10. We have, $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = 1$ ...(i)

Putting $x = \sin\theta$ and $y = \sin\phi$ in (i), we get

$\sin\theta\cos\phi + \cos\theta\sin\phi = 1$

$\Rightarrow$ $\sin(\theta + \phi) = 1 \Rightarrow (\theta + \phi) = \sin^{-1}(1)$

$\Rightarrow \sin^{-1} x + \sin^{-1} y = \dfrac{\pi}{2}$ ...(ii)

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}.$$

11. Let $u = e^x \sin x^3$ and $v = (\tan x)^x$

Now, $u = e^x \sin x^3$

Differentiating w.r.t. x, we get

$$\frac{du}{dx} = e^x \cdot \frac{d\{\sin(x)^3\}}{dx} + \sin x^3 \cdot \frac{d}{dx}(e^x)$$

$$= e^x \cdot \cos x^3 \cdot 3x^2 + \sin x^3 \cdot e^x$$

Hence, $\dfrac{du}{dx} = 3x^2 \cdot e^x \cos x^3 + e^x \sin x^3$

Again, $v = (\tan x)^x \therefore \log v = x \log (\tan x)$

Differentiating w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = 1 \cdot \log(\tan x) + x \cdot \frac{1}{\tan x}\sec^2 x$$

$$\therefore \frac{dv}{dx} = v\,[\log(\tan x) + x \cot x \cdot \sec^2 x]$$

$$= (\tan x)^x [\log (\tan x) + x \cot x \sec^2 x]$$

Now, $y = u + v \Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx} = 3x^2 e^x \cos(x^3) + e^x \sin(x^3) + (\tan x)^x [\log(\tan x) + x \cot x \sec^2 x]$$

12. $x = 3 \sin t - \sin 3t \Rightarrow \dfrac{dx}{dt} = 3\cos t - 3\cos 3t$ ...(i)

$y = 3\cos t - \cos 3t \Rightarrow \dfrac{dy}{dt} = -3\sin t + 3\sin 3t$ ...(ii)

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin 3t - \sin t}{\cos t - \cos 3t} \text{[Dividing (ii) by (i)]}$$

$$= \frac{2\cos 2t \sin t}{2\sin 2t \sin t} = \cot(2t)$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -2\operatorname{cosec}^2 2t \cdot \frac{dt}{dx}$$

$$= -2\operatorname{cosec}^2 2t \cdot \frac{1}{3(\cos t - \cos 3t)} \text{ [From (i)]}$$

At $t = \dfrac{\pi}{3}, \dfrac{d^2y}{dx^2} = -2\operatorname{cosec}^2 \dfrac{2\pi}{3} \cdot \dfrac{1}{3\left(\cos\dfrac{\pi}{3} - \cos\dfrac{3\pi}{3}\right)}$

$$= -2\left(\frac{2}{\sqrt{3}}\right)^2 \cdot \frac{1}{3\left(\frac{1}{2}+1\right)} = -\frac{16}{27}$$

13. Given $f(x) = x(x-1)(x-2) = x(x^2 - 3x + 2)$ ...(i)

$\therefore f'(x) = 1.(x^2 - 3x + 2) + x(2x - 3)$

$= 3x^2 - 6x + 2$ ...(ii)

Clearly, $f'(x)$ is finite and unique for all x and hence $f(x)$ is differentiable as well as continuous for all x.

Hence, $f(x)$ is continuous in $\left[0, \dfrac{1}{2}\right]$.

Also, $f(x)$ is differentiable in $\left(0, \dfrac{1}{2}\right)$

Hence all conditions of Lagrange's mean value theorem are satisfied for $f(x)$ in $\left[0, \dfrac{1}{2}\right]$.

From (i), $f(0) = 0, f\left(\dfrac{1}{2}\right) = \dfrac{1}{2}\left(\dfrac{1}{2}-1\right)\left(\dfrac{1}{2}-2\right) = \dfrac{3}{8}$

Now, $f'(c) = \dfrac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0}$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{\frac{3}{8} - 0}{\frac{1}{2}} = \frac{3}{4}$$

$$\Rightarrow 12c^2 - 24c + 8 = 3 \Rightarrow 12c^2 - 24c + 5 = 0$$

$$\therefore c = \frac{24 \pm \sqrt{576 - 240}}{24} = 1 \pm \frac{\sqrt{21}}{6}$$

Hence, $c = 1 + \dfrac{\sqrt{21}}{6}, 1 - \dfrac{\sqrt{21}}{6}$

But $0 < c < \dfrac{1}{2} \quad \therefore c = 1 - \dfrac{\sqrt{21}}{6}$

Thus, there exists at least one c $\left(= 1 - \dfrac{\sqrt{21}}{6}\right)$ in $\left(0, \dfrac{1}{2}\right)$ such that $f'(c) = \dfrac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0}$

Thus, Lagrange's mean value theorem has been verified.

14. (i) We have, $y = b \tan^{-1}\left(\dfrac{x}{a} + \tan^{-1}\dfrac{y}{x}\right)$

$$\Rightarrow \frac{y}{b} = \tan^{-1}\left(\frac{x}{a} + \tan^{-1}\frac{y}{x}\right)$$

$$\Rightarrow \tan\frac{y}{b} = \frac{x}{a} + \tan^{-1}\frac{y}{x}$$

Differentiating both sides with respect to x, we get

$$\frac{1}{b}\sec^2\frac{y}{b}\frac{dy}{dx}=\frac{1}{a}+\frac{1}{1+(y/x)^2}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{1}{b}\sec^2\frac{y}{b}\frac{dy}{dx}=\frac{1}{a}+\frac{x^2}{x^2+y^2}\frac{x\frac{dy}{dx}-y(1)}{x^2}$$

$$\Rightarrow \frac{1}{b}\sec^2\frac{y}{b}\frac{dy}{dx}=\frac{1}{a}+\frac{x}{x^2+y^2}\frac{dy}{dx}-\frac{y}{x^2+y^2}$$

$$\Rightarrow \left(\frac{1}{b}\sec^2\frac{y}{b}-\frac{x}{x^2+y^2}\right)\frac{dy}{dx}=\frac{1}{a}-\frac{y}{x^2+y^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\frac{1}{a}-\frac{y}{x^2+y^2}}{\frac{1}{b}\sec^2\frac{y}{b}-\frac{x}{x^2+y^2}}$$

(ii) We have, $\sqrt{x}+\sqrt{y}=4$ ...(i)

Differentiating both sides with respect to x, we get

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0\Rightarrow\frac{1}{2\sqrt{y}}\frac{dy}{dx}=\frac{-1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$

Putting $x = 1$ in (i), we get

$$\sqrt{1}+\sqrt{y}=4 \Rightarrow \sqrt{y}=4-1=3\Rightarrow y=9$$

$$\therefore \left.\frac{dy}{dx}\right|_{(1,9)}=-\sqrt{\frac{9}{1}}=-3.$$

15. (i) We have

$f(x) = x^3 + bx^2 + ax + 5, x \in [1, 3]$

$\Rightarrow f'(x) = 3x^2 + 2bx + a$

Since Rolle's theorem holds for the function,

$\therefore f'(c) = 0 \Rightarrow 3c^2 + 2bc + a = 0$

$$\Rightarrow c=\frac{-2b\pm\sqrt{4b^2-12a}}{6}=\frac{-b\pm\sqrt{b^2-3a}}{3}$$

$$\Rightarrow 2+\frac{1}{\sqrt{3}}=\frac{-b\pm\sqrt{b^2-3a}}{3}$$

$$\Rightarrow 2+\frac{1}{\sqrt{3}}=-\frac{b}{3}+\frac{\sqrt{b^2-3a}}{3}$$

$$\Rightarrow 2=-\frac{b}{3} \text{ and } \frac{\sqrt{b^2-3a}}{3}=\frac{1}{\sqrt{3}}$$

$\Rightarrow b = -6$ and $b^2 - 3a = 3$

$\Rightarrow b = -6$ and $a = 11$

(ii) Let $f(x) = x^2$

(a) $f(x) = x^2$, being a polynomial, is a continuous function on $[-2, 2]$.

(b) $f'(x) = 2x$ which exists in $(-2, 2)$

$\therefore f(x)$ is derivable in $(-2\ 2)$.

(c) Also, $f(-2) = f(2) = 4$

Thus, all the conditions of Rolle's theorem are satisfied. Hence there must exist at least one value $c \in (-2, 2)$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Rightarrow 2c = 0$ [$\because f'(x) = 2x$]

$\Rightarrow c = 0 \in (-2, 2)$

Thus, the tangent to the curve is parallel to x-axis at $x = 0$.

At $x = 0, y = 0$. [$\because y = x^2$]

$\therefore$ Tangent to the curve $y = x^2$ is parallel to x-axis at (0,0).