COMPLEX NUMBERS

INTRODUCTION

For the complex number $z = a + ib$, a is called the real part, denoted by Re(z) and b is called the imaginary part denoted by Im(z). A complex number z is said to be real if Im(z) = 0 and is said to be purely imaginary if Re(z) = 0. Thus the complex number $0 = 0 + 0i$, is both real and purely imaginary. $a + ib$ is called an imaginary number if $b \neq 0$.

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if and only if $a = c$ and $b = d$. However there is no order relation in the set of complex numbers *i.e.*, the expressions of the form $a + ib < c + id$ and $a + ib > c + id$ are meaningless unless $b = d = 0$.

ALGEBRA OF COMPLEX NUMBERS

Let $a, b, c, d, \lambda \in R$, then

(i) Addition : $(a + ib) + (c + id) = (a + c) + i(b + d)$

(ii) Subtraction : $(a + ib) - (c + id) = (a - c) + i(b - d)$

(iii) Multiplication :

$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$

(iv) Division :

For $c+id \neq 0, \dfrac{a+ib}{c+id} = \dfrac{ac+bd}{c^2+d^2} + i\dfrac{bc-ad}{c^2+d^2}$

(v) Multiplication by a real number :

$\lambda(a + ib) = \lambda a + i(\lambda b)$

THE CONJUGATE OF A COMPLEX NUMBER

Let $z = a + ib$ be a complex number. The conjugate of z, denoted as $\bar{z}$, is the complex number $a - ib$, *i.e.*, $\bar{z} = a - ib$.

Thus $a = \text{Re}(z) = \dfrac{z+\bar{z}}{2}$ and $b = \text{Im}(z) = \dfrac{z-\bar{z}}{2i}$

Let z, z_1 and z_2 be complex numbers. The following results can easily be proved.

(i) $z = \bar{z} \Leftrightarrow z$ is a real number.

(ii) $z = -\bar{z} \Leftrightarrow z$ is a purely imaginary number.

(iii) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$ **(iv)** $\overline{z_1-z_2} = \bar{z}_1 - \bar{z}_2$

(v) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ **(vi)** $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

(vii) $\overline{(\bar{z})} = z$ **(viii)** $\overline{(z^n)} = (\bar{z})^n, n \in I$

(ix) $\overline{f(z)} = f(\bar{z})$, f being a polynomial with real coefficients.

(x) Let $z = x + iy$, then $z\bar{z} = x^2 + y^2$. Thus $z\bar{z}$ is a non-negative real number for any complex number.

NOTE : The imaginary roots of a polynomial equation over reals, occur in conjugate pairs and hence any polynomial equation of odd degree over reals has at least one real root. Obviously this is not the case for the polynomial equations over C, the set of complex numbers.

THE MODULUS OF A COMPLEX NUMBER

Let $z = a + ib$ be a complex number. The modulus of z, denoted by $|z|$, is defined to be a non-negative real number $\sqrt{a^2+b^2}$, *i.e.* $|z| = \sqrt{a^2+b^2}$.

Alternate definition :

Let z be a complex number then $|z|$ can be defined as $(z\bar{z})^{1/2}$.

Let z, z_1 and z_2 be complex numbers, then

(i) $|z| = |\bar{z}| \geq 0$ and equality holds if and only if $z = 0$.

(ii) $\text{Re}(z) \leq |z|$ and equality holds if and only if $\text{Im}(z) = 0$ and $\text{Re}(z) \geq 0$.

(iii) $\text{Im}(z) \leq |z|$ and equality holds if and only if $\text{Re}(z) = 0$ and $\text{Im}(z) \geq 0$.

(iv) $z\bar{z} = |z|^2$ and hence $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$, $z \neq 0$

(v) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$

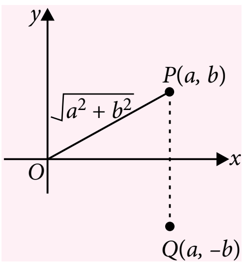
(vi) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\bar{z}_2)$

(vii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(viii) $|z_1 z_2| = |z_1| \cdot |z_2|$

(ix) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS



A complex number $a + ib$ corresponds to the ordered pair (a, b), which can be represented geometrically as the unique point (a, b) in the xy plane and vice-versa.

The xy plane having a complex number assigned to each of its points is called the complex plane or the Argand plane.

Let $z = a + ib$. Then z is represented as the point $P(a, b)$ on the argand plane. Obviously $OP = \sqrt{a^2 + b^2} = |z|$. Hence $|z|$ is the distance of z from origin. Further if $Q(a, -b)$ is the mirror image of $P(a, b)$ in the x-axis (also called as real axis), then Q is the geometrical representation of $\bar{z}$. Also note that a complex number is real if and only if it lies on the real axis and a complex number is purely imaginary if and only if it lies on the imaginary axis. For any complex number z_1 and z_2, it is easy to see that $|z_1 - z_2|$ is the distance of z_1 from z_2, when z_1 and z_2 are represented as points on the argand plane.

NOTE : The locus of z satisfying $|z| = r$, $r > 0$ is the circle having centre at origin and radius r. Similarly, locus of z satisfying $|z - z_1| = r$, $r > 0$ is the circle having centre at z_1 and radius r.

The Geometrical representation of $z_1 + z_2$ and $z_1 - z_2$: Let P, Q represent two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ respectively in the Argand plane. Join the origin O with the points P and Q and complete the parallelogram $OPRQ$. It is clear that the coordinates of the point R are $(x_1 + x_2, y_1 + y_2)$ and hence the point R represents the complex number $(x_1 + x_2) + i(y_1 + y_2) = z_1 + z_2$. In triangle OPR, OR = distance of R from O = $|z_1 + z_2|$. Similarly $OP = |z_1|$ and $PR = OQ = |z_2|$. Using the triangle inequality in triangle OPR, we get $OR < OP + PR \Rightarrow |z_1 + z_2| < |z_1| + |z_2|$. However if O, P and Q are collinear (P and Q being on the same side of O), then $OR = OP + PR$, *i.e.* $|z_1 + z_2| = |z_1| + |z_2|$.

Hence for any complex numbers z_1, z_2; $|z_1 + z_2| \leq |z_1| + |z_2|$. This is called the triangle inequality.

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Its geometrical interpretation can now be explained as $OR^2 + PQ^2 = OP^2 + PR^2 + RQ^2 + QO^2$, in a parallelogram the sum of the squares of the diagonals is equal to the sum of the squares of the four sides.

ARGUMENT OF A COMPLEX NUMBER

Let z be a non-zero complex number, which corresponds to the point P on the Argand plane. If $|z| = r$, then θ satisfies $x = r\cos\theta$, $y = r\sin\theta$, where $z = x + iy$, and θ is called the argument of the given complex number.

The angle of the line segment OP with the positive direction of x-axis is θ. The angle can also be taken as $2n\pi + \theta$, $n \in I$. This angle is called argument or amplitude of the complex number z denoted by arg(z) or amp(z). Value of θ having minimum modulus value is called the principal argument of z.

Thus for any non-zero $z \in C$, principal arg(z) $\in (-\pi, \pi]$. Obviously a non-zero complex number z is real if and only if principal argument of z is 0 or π and it is purely imaginary if and only if principal argument of z is $\pm\frac{\pi}{2}$.

Some Properties of Argument

Let z be a non-zero complex number and $n \in I$

(i) $\arg(-z) = \arg(z) + (2n + 1)\pi$

(ii) $\arg(z+\bar{z}) = n\pi$, z not being purely imaginary.

(iii) $\arg(z-\bar{z}) = (4n\pm1)\dfrac{\pi}{2}$, z not being purely real.

(iv) $\arg\bar{z} = \arg\left(\dfrac{z\bar{z}}{z}\right) = \arg\left(\dfrac{1}{z}\right) + 2n\pi$

(v) $\arg z + \arg\bar{z} = 2n\pi$

Method of finding principal argument of a non-zero complex number

Let $z = x + iy$ be a given non-zero complex number, whose principal argument is to be found.

Let $\alpha = \tan^{-1}\left|\dfrac{y}{x}\right|$. Now α, $\pi - \alpha$, $-\pi + \alpha$, $-\alpha$ becomes the principal argument of z according as the point $P(z)$ lies in first, second, third, fourth quadrant respectively.

POLAR FORM OF A COMPLEX NUMBER

Let z be a given non-zero complex number, which corresponds to the point P in the Argand plane. Let $z = x + iy$. Then $OM = x$, $PM = y$. Let $r = |z|$, then $OP = r$. Let $\theta = \arg(z)$, then $\angle POM = \theta$.
Now $x = r\cos\theta$ and $y = r\sin\theta$.
Hence $z = x + iy = r(\cos\theta + i\sin\theta)$. This form is called polar form of z, sometimes written in short as $r\text{cis}\theta$. Note that (r, θ) are polar coordinates of z.

NOTE : It is a known fact that

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \ldots\ldots$$

$$\text{and } \sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} + \ldots\ldots$$

$$\text{Hence } \cos\theta + i\sin\theta = 1 + \frac{i\theta}{\lfloor 1} + \frac{(i\theta)^2}{\lfloor 2} + \frac{(i\theta)^3}{\lfloor 3} + \ldots\ldots$$

$$\text{Also we know that } e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots$$

Hence, $e^{i\theta} = \cos\theta + i\sin\theta$

Some Consequences of Euler Formula

- Let z be a non-zero complex number and let $re^{i\theta}$ be its polar form. As $\arg(z) = \theta$, so
  $\arg(z) = 2n\pi + \theta$, $n \in I$. Hence $z = re^{i(\theta+2n\pi)}$.

Now $\log z = \log re^{i(\theta+2n\pi)} = \log r + i(\theta + 2n\pi), n \in I.$

Thus logarithm of an imaginary number is not unique.

- $\cos\theta + i\sin\theta = e^{i\theta} \Rightarrow \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$

Thus, $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

DE MOIVRE'S THEOREM

We define $e^{i\theta} = \cos\theta + i\sin\theta$.
Let n be any integer, then

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

Thus $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, $n \in I$
This result is called De Moivre's Theorem for an integer. However if $n = \dfrac{p}{q}, p, q \in I, p \neq 0$ and $q > 1$, then

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{p/q} = \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q},$$

which is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$.
To make this point more clear, consider $(-1)^{1/2}$ *i.e.*, all the complex numbers whose square is -1, which are i and $-i$, for $-1 = \cos\pi + i\sin\pi$. Hence

$$(-1)^{1/2} = (\cos\pi + i\sin\pi)^{1/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i,$$

which is one of the value of $(-1)^{1/2}$.
For the other value, we write $-1 = \cos 3\pi + i\sin 3\pi$

$$\Rightarrow \quad (-1)^{1/2} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$

NOTE : De Moivre's Theorem is true for irrational values of n also.

Applications of De Moivre's Theorem to Trigonometry

(i) Let n be a positive integer. Then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. Also expanding $(\cos\theta + i\sin\theta)^n$ by using Binomial Theorem and comparing the real and imaginary parts, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots.$$

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots.$$

$$\text{Hence, } \tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + \ldots\ldots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \ldots\ldots}$$

(ii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1z_2 = r_1e^{i\theta_1}\cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)} = r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

Hence $|z_1z_2| = r_1r_2 = |z_1|\,|z_2|$
and $\arg(z_1z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

NOTE : (i) Infact, $\arg(z_1z_2) = \arg z_1 + \arg z_2 + 2n\pi$, for some $n \in I$

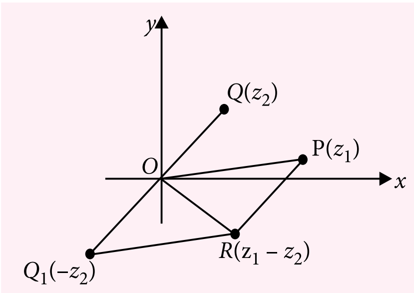
Similarly, $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 + 2n\pi$, for some $n \in I$

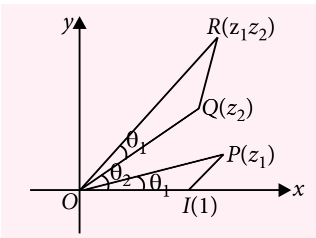
and $\arg(z^n) = n\arg(z) + 2k\pi$, $k \in I$

(ii) Let $z = re^{i\alpha}$. Then $ze^{i\theta} = re^{i(\alpha+\theta)}$. Thus the complex number $ze^{i\theta}$ (in the Argand plane) can be obtained from z by rotating z about origin by an angle θ.

GEOMETRICAL REPRESENTATION OF z_1z_2 AND z_1/z_2

Let P and Q represent two complex numbers z_1 and z_2 respectively in the Argand plane. Join the origin O with the points P and Q. Now we are in search of a point $R(z_1z_2)$ in the Argand plane. For that let I be the point representing the number 1. Now consider a triangle OQR, which is similar to the triangle OIP.





Now argument of the complex number corresponding to R is $\theta_1 + \theta_2 = \arg(z_1z_2)$

Further

$$\frac{OR}{OP} = \frac{OQ}{OI} \Rightarrow OR = \frac{|z_1|\cdot|z_2|}{1}$$

Thus the complex number corresponding to R is $|z_1|.|z_2|.\ (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

$= |z_1|(\cos\theta_1 + i\sin\theta_1)\ |z_2|(\cos\theta_2 + i\sin\theta_2) = z_1z_2$

For the representation of $\dfrac{z_1}{z_2}$ in the Argand plane once again, let P and Q represent the complex numbers z_1 and z_2 in the Argand plane (as shown in the figure). Now construct a triangle OPR similar to triangle OQI. Then argument of the complex number corresponding to R is

$$\theta_1 - \theta_2 = \arg\left(\frac{z_1}{z_2}\right)$$

Further $\dfrac{OR}{OP} = \dfrac{OI}{OQ} \Rightarrow OR = \dfrac{OP}{OQ} = \dfrac{|z_1|}{|z_2|} = \left|\dfrac{z_1}{z_2}\right|$

Hence the complex number corresponding to R is

$$\frac{|z_1|}{|z_2|}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$
$$= \frac{|z_1|(\cos\theta_1 + i\sin\theta_1)}{|z_2|(\cos\theta_2 + i\sin\theta_2)} = \frac{z_1}{z_2}$$

CONCEPT OF ROTATION

Let z_1, z_2 and z_3 be the vertices of a triangle ABC described in the anticlockwise sense.

$AB = |z_2 - z_1|$, $AC = |z_3 - z_1|$ and

$BC = |z_3 - z_2| = |(z_3 - z_1) - (z_2 - z_1)|$

Now $\arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) = \arg(z_3 - z_1) - \arg(z_2 - z_1) = \alpha$

Also $\left|\dfrac{z_3 - z_1}{z_2 - z_1}\right| = \dfrac{AC}{AB}$

Therefore the polar form of $\dfrac{z_3 - z_1}{z_2 - z_1}$ is $\dfrac{AC}{AB}(\cos\alpha + i\sin\alpha)$

if $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$, then the triangle ABC is equilateral.

CONDITION FOR THE FOUR POINTS TO BE CONCYCLIC

Let z_1, z_2, z_3 and z_4 be four concyclic points, represented by A, B, C and D in the Argand plane.

As the points A, B, C and D are concyclic,

$$\angle ADB = \angle ACB \Rightarrow \arg\left(\frac{z_2 - z_4}{z_1 - z_4}\right) = \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$$

(using the concept of rotation)

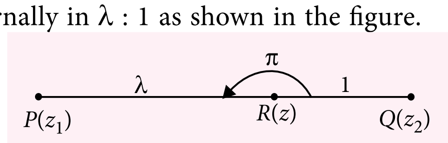
$$\Rightarrow \arg\left(\frac{z_2 - z_4}{z_1 - z_4} \times \frac{z_1 - z_3}{z_2 - z_3}\right) = 0$$

$\Rightarrow \dfrac{(z_2 - z_4)(z_1 - z_3)}{(z_1 - z_4)(z_2 - z_3)}$ is a positive real number.

NOTE : The converse of the above result is also true, *i.e.*, if a complex number of the form $\dfrac{(z_2 - z_4)(z_1 - z_3)}{(z_1 - z_4)(z_2 - z_3)}$ is a positive real number, then the points represented by z_1, z_2, z_3 and z_4 on the Argand plane are concyclic.

THE SECTION FORMULA

Let $P(z_1)$ and $Q(z_2)$ be two given points. Let $R(z)$ be the point which divides the join of P and Q internally in $\lambda : 1$ as shown in the figure.



Using the concept of rotation at the point R, we get

$$\frac{z-z_1}{z-z_2}=\lambda e^{i\pi} \Rightarrow z=\frac{z_1+\lambda z_2}{1+\lambda}$$

If R divides PQ externally in $\lambda : 1$, then it can be shown that $z=\frac{z_1-\lambda z_2}{1-\lambda}$.

THE n^{th} ROOTS OF UNITY

Consider the polynomial equation $z^n = 1$.

The roots of this equation are n in number and are called n^{th} roots of unity.

In order to find the roots of $z^n = 1$, we write the polar form of 1, *i.e.*,

$$z^n = \cos 2k\pi + i \sin 2k\pi,\ k \in I$$

$$\Rightarrow z=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}=e^{\frac{2k\pi i}{n}},\ k\in I$$

Now if we give different integral values to k, then we always get roots of $z^n = 1$, but some of them may be repeated. However, if we take $k = 0, 1, 2, 3, ..., n - 1$, then we get (all) distinct roots of $z^n = 1$.

Thus n, n^{th} roots of unity are

$e^{\frac{2k\pi i}{n}}$, $k = 0, 1, 2, ..., n - 1$.

If we take $\alpha=e^{\frac{2\pi i}{n}}$, then the n^{th} roots of unity are $1, \alpha, \alpha^2, ..., \alpha^{n-1}$.

PROPERTIES OF n, n^{th} ROOTS OF UNITY

(i) If n, n^{th} roots of unity are represented on the argand plane, then they form the vertices of a regular n-gon having centre at origin and circumcircle of this n-gon is $|z| = 1$, as modulus value of every root is 1.

(ii) $1 + \alpha + \alpha^2 + ... + \alpha^{n-1} = 0$

(iii) $1 + \alpha^p + (\alpha^p)^2 + ... + (\alpha^p)^{n-1} = 0$, if g.c.d. $(p, n) = 1$.

(iv) $1\cdot\alpha\cdot\alpha^2 ...... \alpha^{n-1} = (-1)^{n-1}$

(v) $z^n - 1 \equiv (z - 1)(z - \alpha)(z - \alpha)^2 ..... (z - \alpha^{n-1})$

(vi) The imaginary roots form conjugate pair.

EQUATION OF STRAIGHT LINE IN COMPLEX FORM

(i) Equation of line through the points z_1 and z_2 in parametric form is given by

$$z = z_2 + \lambda(z_1 - z_2),\ \lambda \in R$$

(ii) Equation of line through z_1 and z_2 is given by

$$\frac{z-z_1}{z_2-z_1}=\frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1} \text{ or } \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix}=0$$

(iii) General equation of a line is given by $\bar{a}z+a\bar{z}+b=0$, where a is a complex number and b is a real number. Slope of this line is given by $-\frac{\text{Re}(a)}{\text{Im}(a)}$ and its distance from a point z_1 is given by $\left|\frac{\bar{a}z_1+a\bar{z}_1+b}{2|a|}\right|$.

COMPLEX SLOPE OF A LINE

If a line passes through the points z_1 and z_2 then its complex slope is defined as $\frac{z_1-z_2}{\bar{z}_1-\bar{z}_2}$. The complex slope of the line $\bar{a}z+a\bar{z}$ is given by $-\frac{a}{\bar{a}}$.

EQUATION OF A CIRCLE IN COMPLEX FORM

(i) The equation of the circle having centre at z_o and radius r is given by $|z - z_o| = r$

(ii) The general equation of circle is $z\bar{z}+\bar{a}z+a\bar{z}+b=0$, where a is a complex number and b is a real number. The centre of this circle is '$-a$' and radius is $\sqrt{|a|^2-b}$.

(iii) The equation of the circle described on the line segment joining z_1 and z_2 as diameter is given by $(z-z_1)(\bar{z}-\bar{z}_2)+(z-z_2)(\bar{z}-\bar{z}_1)=0$.

(iv) $\left|\frac{z-z_1}{z-z_2}\right|=\lambda,\ \lambda>0,\ \lambda\neq 1$ represents a circle having diameter AB, when A and B divide the join of z_1 and z_2 in $\lambda : 1$ internally and externally respectively.

(v) $\arg\left(\frac{z-z_1}{z-z_2}\right)=\theta$, represents an arc of the circle through z_1 and z_2.

(vi) $|z - z_1|^2 + |z - z_2|^2 = k$, represents a circle provided $k>\frac{1}{2}|z_1-z_2|^2$.

EQUATION OF CONIC SECTION IN COMPLEX FORM

(i) Equation of parabola with focus at z_1 and directrix $a\bar{z}+\bar{a}z+b=0$ is $|z-z_1|=\left|\frac{a\bar{z}+\bar{a}z+b}{2|a|}\right|$.

(ii) Equation of ellipse with foci at z_1 and z_2 and length of major axis λ, is $|z - z_1| + |z - z_2| = \lambda$.

(iii) Equation of hyperbola with foci at z_1 and z_2 and length of transverse axis λ, is $||z - z_1| - |z - z_2|| = \lambda$.

PROBLEMS

SECTION-I

Single Correct Answer Type

1. If $n \geq 3$ and $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are n roots of unity, then value of $\sum\limits_{1 \le i < j \le n-1} \alpha_i \alpha_j$ is

(a) 0 (b) 1 (c) −1 (d) $(-1)^n$

2. Let $z = \cos\theta + i\sin\theta$. Then, the value of $\sum\limits_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is

(a) $\frac{1}{2°}$ (b) $\frac{1}{3\sin 2°}$ (c) $\frac{1}{2\sin 2°}$ (d) $\frac{1}{4\sin 2°}$

3. If the complex number z satisfying $z + |z| = 2 + 8i$ then value of $|z| =$

(a) 8 (b) 17 (c) 15 (d) 24

4. If $|z + 2 - i| = 5$ then maximum value of $|3z + 9 - 7i| =$

(a) 20 (b) 15 (c) 5 (d) 16

5. If $\lambda \in R$ and non real roots of $2z^2 + 2z + \lambda = 0$ and (0, 0) forms vertices of an equilateral triangle, then λ =

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

6. If z and w are two complex numbers such that $\bar{z} + i\bar{w} = 0$ and $\arg(zw) = \pi$, then $\arg(z)$ =

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) $\frac{5\pi}{4}$

7. If $A(z_1), B(z_2), C(z_3)$ are vertices of a triangle such that $z_3 = \left(\frac{z_2 - iz_1}{1-i}\right)$ and $|z_1| = 3, |z_2| = 4$ and $|z_2 + iz_1| = |z_1| + |z_2|$, then area of triangle ABC is

(a) $\frac{5}{2}$ (b) 0 (c) $\frac{25}{2}$ (d) $\frac{25}{4}$

8. The radius of the circle given by $\arg\left(\frac{z-5+4i}{z+3-2i}\right) = \frac{\pi}{4}$ is

(a) $5\sqrt{2}$ (b) 5 (c) $\frac{5}{\sqrt{2}}$ (d) $\sqrt{2}$

9. If $f(x) = 2x^3 + 2x^2 - 7x + 72$ then $f\left(\frac{3-5i}{2}\right) =$

(a) 1 (b) 2 (c) 3 (d) 4

10. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$ then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma =$

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) 4 (d) 1

11. If z_1 and z_2 are two complex numbers such that $z_1^2 + z_2^2 \in R$ and $z_1(z_1^2 - 3z_2^2) = 2$, $z_2(3z_1^2 - z_2^2) = 11$, then $z_1^2 + z_2^2 =$

(a) 5 (b) 125 (c) 25 (d) 15

12. If z is a non-real complex number, then the minimum value of $\frac{\text{Im}\, z^5}{\text{Im}^5 z}$ is

(a) −1 (b) −2 (c) −4 (d) −5

13. Let $z_r (1 \le r \le 4)$ be complex numbers such that $|z_r| = \sqrt{r+1}$ and $|30z_1 + 20z_2 + 15z_3 + 12z_4|$
$= k|z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_1 + z_4z_1z_2|$
Then the value of k equals

(a) $|z_1z_2z_3|$ (b) $|z_2z_3z_4|$

(c) $|z_4z_1z_2|$ (d) None of these

14. If P and Q are represented by the complex numbers z_1 and z_2 such that $\left|\frac{1}{z_2} + \frac{1}{z_1}\right| = \left|\frac{1}{z_2} - \frac{1}{z_1}\right|$, then the circumcentre of ΔOPQ, (where O is the origin) is

(a) $\frac{z_1 - z_2}{2}$ (b) $\frac{z_1 + z_2}{2}$ (c) $\frac{z_1 + z_2}{3}$ (d) $z_1 + z_2$

15. If α is non real root of $x^7 = 1$, then $1 + 3\alpha + 5\alpha^2 + 7\alpha^3 + \ldots + 13\alpha^6$ is equal to

(a) 0 (b) $\frac{14}{1-\alpha}$ (c) $\frac{14}{\alpha - 1}$ (d) none of these

16. If z_1, z_2 are two complex numbers satisfying the equation $\left|\frac{z_1 + z_2}{z_1 - z_2}\right| = 1$, then $\frac{z_1}{z_2}$ is a number which is

(a) Positive real (b) Negative real

(c) Zero (d) Lying on imaginary axis

17. If z_1, z_2 and z_3 are the vertices of ΔABC, which is not right angled triangle taken in anti-clockwise direction and z_0 is the circumcentre, then $\left(\frac{z_0 - z_1}{z_0 - z_2}\right)\frac{\sin 2A}{\sin 2B} + \left(\frac{z_0 - z_3}{z_0 - z_2}\right)\frac{\sin 2C}{\sin 2B}$ is equal to

(a) 0 (b) 1

(c) −1 (d) 2

18. If 'a' is a complex number such that $|a| = 1$, then the values of 'a' so that equation $az^2 + z + 1 = 0$ has one purely imaginary root is

(a) $a = \cos\theta + i\sin\theta, \theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$

(b) $a = \sin\theta + i\cos\theta, \theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$

(c) $a = \cos\theta + i\sin\theta, \theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{4}\right)$

(d) $a = \sin\theta + i\cos\theta, \theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{4}\right)$

19. Let $A(z_1)$, $B(z_2)$, $C(z_3)$ be the vertices of a triangle oriented in anti-clockwise direction. If $BC : CA : AB = 2 : \sqrt{2} : \sqrt{3}+1$, then the imaginary part of $\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^4$ is

(a) 0 (b) $-7+2\sqrt{6}$
(c) $7-2\sqrt{6}$ (d) cannot be determined

20. If z is a complex number having least absolute value and $|z - 2 + 2i| = 1$, then $z =$

(a) $\left(2-\frac{1}{\sqrt{2}}\right)(1-i)$ (b) $\left(2-\frac{1}{\sqrt{2}}\right)(1+i)$
(c) $\left(2+\frac{1}{\sqrt{2}}\right)(1-i)$ (d) $\left(2+\frac{1}{\sqrt{2}}\right)(1+i)$

21. Sum of common roots of the equation $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ is

(a) –1 (b) 1 (c) 0 (d) 1

22. Let z be a complex number having the argument θ, $0 < \theta < \frac{\pi}{2}$ and satisfying the equation, $|z - 3i| = 3$. Then $\cot\theta - \frac{6}{z} =$

(a) i (b) $-i$ (c) $2i$ (d) $-2i$

23. Let z_1 and z_2 be any two complex numbers then $\left|z_1+\sqrt{z_1^2-z_2^2}\right|+\left|z_1-\sqrt{z_1^2-z_2^2}\right|$ is equal to

(a) $|z_1^2 - z_2^2| + |z_1^2 + z_2^2|$ (b) $|z_1 - z_2| + |z_1^2 + z_2^2|$
(c) $|z_1 + z_2| + |z_1^2 + z_2^2|$ (d) $|z_1 + z_2| + |z_1 - z_2|$

24. Both the roots of the equation $z^2 + az + b = 0$ are of unit modulus if

(a) $|a| \le 2$, $|b| = 1$, $\arg b = 2 \arg a$
(b) $|a| \le 2$, $|b| = 1$, $\arg b = \arg a$
(c) $|a| \ge 2$, $|b| = 2$, $\arg b = 2 \arg a$
(d) $|a| \ge 2$, $|b| = 2$, $\arg b = \arg a$

25. If $|z - i| = 1$ and $\arg(z) = \theta$ where $\theta \in \left(0, \frac{\pi}{2}\right)$, then $\cot\theta - \frac{2}{z}$ equals

(a) $2i$ (b) $3i$ (c) i (d) $-i$

SECTION-II

Multiple Correct Answer Type

26. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\text{Re}(z_1\bar{z}_2) = 0$ then the pair of complex numbers $\omega_1 = a + ic$ and $\omega_2 = b + id$ satisfies

(a) $|\omega_1| = 1$ (b) $|\omega_2| = 1$
(c) $\text{Re}(\omega_1\bar{\omega}_2) = 0$ (d) $\omega_1\bar{\omega}_2 = 0$

27. If points A and B are represented by the non-zero complex numbers z_1 and z_2 on the Argand plane such that $|z_1 + z_2| = |z_1 - z_2|$ and O is the origin, then

(a) orthocentre of ΔOAB lies at O
(b) circumcentre of ΔOAB is $\frac{z_1 + z_2}{2}$
(c) $\text{Arg}\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2}$
(d) ΔOAB is isosceles

28. The adjacent vertices of a regular polygon of n sides whose centre is at origin are given by $(1+\sqrt{2}, 1)$, $(1+\sqrt{2}, -1)$. Then the value of n is

(a) 8 (b) 4 (c) 12 (d) 6

29. If α is a variable complex number such that $|\alpha| > 1$ and $z = \alpha + \frac{1}{\alpha}$ lies on a conic then

(a) Eccentricity of the conic is $\frac{2|\alpha|}{1+|\alpha|^2}$
(b) Distance between foci is 4
(c) Length of latus rectum is $\frac{2(|\alpha|^2 - 1)}{|\alpha|^2 + 1}$
(d) Distance between directrix is $\left(|\alpha| + \frac{1}{|\alpha|}\right)^2$

30. If $z = 1 + \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$ then

(a) $|z| = 2\cos\frac{3\pi}{5}$ (b) $|z| = 2\cos\frac{2\pi}{5}$
(c) $\arg z = \frac{3\pi}{5}$ (d) $\arg z = -\frac{2\pi}{5}$

31. Let $z_1, z_2, z_3, \ldots, z_n$ are the complex numbers such that $|z_1| = |z_2| = \ldots = |z_n| = 1$. If $z = \left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right)$ then

(a) z is purely imaginary
(b) z is real
(c) $0 < z \le n^2$
(d) z is a complex number of the form $a + ib$

32. If a, b, c are non – zero complex numbers of equal moduli and satisfy $az^2 + bz + c = 0$ then

(a) $\min|z| = \frac{\sqrt{5}-1}{2}$ (b) $\min |z| = 0$
(c) $\min |z|$ does not exist (d) $\max|z| = \frac{\sqrt{5}+1}{2}$

33. If $|z - 3| = \min\{|z - 1|, |z - 5|\}$ then $\text{Re}(z) =$

(a) 2 (b) $\frac{5}{2}$ (c) $\frac{7}{2}$ (d) 4

34. The complex slope μ of a line containing the points z_1 and z_2 in the complex plane is defined as

$\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2}$. If μ_1, μ_2 are the complex slopes of two lines L_1 and L_2, then

(a) L_1 and L_2 are perpendicular if $\mu_1 + \mu_2 = 0$
(b) L_1 and L_2 are parallel if $\mu_1 + \mu_2 = 0$
(c) L_1 and L_2 are perpendicular if $\mu_1\mu_2 = -1$
(d) L_1 and L_2 are parallel if $\mu_1 = \mu_2$

SECTION-III

Comprehension Type

Paragraph for Question No. 35 to 37

Let z_1 be a complex number of magnitude unity and z_2 be a complex number given by $z_2 = z_1^2 - z_1$.

35. If $\arg z_1 = \theta$, then $|z_2|$ is equal to

(a) $2\left|\sin\frac{\theta}{2}\right|$ (b) $2\left|\cos\frac{\theta}{2}\right|$

(c) $\sqrt{2}\left|\sin\frac{\theta}{2}\right|$ (d) $\sqrt{2}\left|\cos\frac{\theta}{2}\right|$

36. If $\arg z_1 = \theta$ and $4n\pi < \theta < (4n + 2)\pi$ (n is an integer), then $\arg z_2$ is equal to

(a) $\frac{3\theta}{2}$ (b) $\frac{\pi - 3\theta}{2}$

(c) $\frac{\pi + 3\theta}{2}$ (d) $\frac{\pi + \theta}{2}$

37. If $\arg z_1 = \theta$ and $(4n + 2)\pi < \theta < (4n + 4)\pi$ (n is an integer), then $\arg z_2$ is equal to

(a) $\frac{\pi}{2} + 3\theta$ (b) $\frac{3\pi}{2} + \frac{3\theta}{2}$

(c) $\frac{3\pi}{2} + 3\theta$ (d) $\frac{\pi}{2} + \frac{3\theta}{2}$

Paragraph for Question No. 38 to 40

Let $A_1, A_2, A_3, ..., A_n$ be a regular polygon of 'n' sides whose centre is origin O. Let the complex numbers representing vertices $A_1, A_2, A_3, ..., A_n$ be $z_1, z_2, z_3, ..., z_n$ respectively. Let $OA_1 = OA_2 = ... = OA_n = 1$

38. The value of $|A_1A_2|^2 + |A_1A_3|^2 + ... + |A_1A_n|^2 =$

(a) n (b) $2n$
(c) $2(n - 1)$ (d) $2(n + 1)$

39. The distances A_1A_j ($j = 2, 3, ..., n$) must be equal to

(a) $\sin\frac{j\pi}{n}$ (b) $2\sin\frac{(j-1)\pi}{n}$

(c) $\cos\frac{j\pi}{n}$ (d) $2\sin\frac{(j+1)\pi}{n}$

40. The value of $|A_1A_2|\,|A_1A_3|\,.....\,|A_1A_n|$ must be equal to

(a) 1 (b) n (c) $\sqrt{n}$ (d) n^2

Paragraph for Question No. 41 to 43

Consider a complex number $w = \frac{z - i}{2z + 1}$ where $z = x + iy$ and $x, y \in R$

41. If the complex number w is purely imaginary then locus of z is

(a) a straight line
(b) a circle with centre $\left(-\frac{1}{4}, \frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{4}$
(c) a circle with centre $\left(\frac{1}{4}, -\frac{1}{2}\right)$ and passing through origin
(d) neither a circle nor a straight line

42. If the complex number w is purely real, then locus of z is

(a) a straight line passing through origin
(b) a straight line with gradient 3 and y intercept (-1)
(c) a straight line with gradient 2 and y intercept 1
(d) a circle

43. If $|w| = 1$, then locus of z is

(a) a point circle
(b) an imaginary circle
(c) a real circle
(d) not a circle

Paragraph for Question No. 44 to 46

If $z = x + iy$ and (x, y) is a point represented by the complex number 'z' in the argand plane. $|z_1 - z_2|$ denotes distance between z_1 and z_2 in the argand plane.

44. The complex number $z = x + iy$ which satisfies the equation $\left|\frac{z - 5i}{z + 5i}\right| = 1$ lies on the

(a) x-axis
(b) y-axis
(c) circle with radius 5 and centre at origin
(d) line $y = 5$

45. If $|z| = 5$ then the points representing the complex number $-i + \frac{15}{z}$ lie on the circle with

(a) centre (0, 1) and radius = 3
(b) centre (0, –1) and radius = 3
(c) centre (1, 0) and radius = 5
(d) centre (–1, 0) and radius = 15

46. If $|z - i| \le 2$ and $z_1 = 3 + 4i$, then the maximum value of $|iz + z_1|$ is

(a) $\sqrt{20} - 2$ (b) 9
(c) $\sqrt{20} + 2$ (d) 8

SECTION-IV

Matrix-Match Type

47. Consider complex number

$z = \cos\alpha + i\sin\alpha, 0 < \alpha < \frac{\pi}{6}$. Then the argument of

Column-I		Column-II	
(A)	$1 + z^3 =$	(p)	$2\alpha - \frac{\pi}{2}$
(B)	$1 - z^4 =$	(q)	$\frac{\pi}{2} + \frac{\alpha}{2}$
(C)	$\frac{1+z^3}{1-z^4} =$	(r)	$\frac{3\alpha}{2}$
(D)	$\frac{z^4-1}{z^3+1} =$	(s)	$\frac{\pi}{2} - \frac{\alpha}{2}$

48. The complex numbers $z_1, z_2, ..., z_n$ represent the vertices of a regular polygon of n sides, inscribed in a circle of unit radius and $z_3 + z_n = Az_1 + \bar{A}z_2$, $[x]$ be the greatest integer $\le x$. Then

When n equals to		$[\lvert A\rvert]$ equals to	
(A)	4	(p)	0
(B)	6	(q)	1
(C)	8	(r)	2
(D)	12	(s)	3

49. Match the following.

Column-I		Column-II	
(A)	The number of integral solutions of the equation $(1 - i)^n = 2^n$ is	(p)	4
(B)	The number of common roots of the equations $x^3 + 2x^2 + 2x + 1 = 0$ and $x^{2000} + x^{2002} + 1 = 0$ is	(q)	3
(C)	The number of all non-zero complex numbers 'z' satisfying $\bar{z} = iz^2$ is	(r)	2
(D)	If z is a complex number, then the number of solutions of $z^2 + \lvert z\rvert = 0$ is	(s)	1

SECTION-V

Integer Answer Type

50. If $\frac{3iz_2}{5z_1}$ is purely real, then find $5\left|\frac{3z_1+7z_2}{3z_1-7z_2}\right|$.

51. If a complex number z satisfies $|z-8-4i| + |z-14-4i| = 10$, then the maximum value of $\arg(z) = \tan^{-1}\frac{11}{3k}$, find k.

52. If 'a' and 'b' are complex numbers. One of the roots of the equation $x^2 + ax + b = 0$ is purely real and the other is purely imaginary then $a^2 - \bar{a}^2 = kb$, find k.

53. The sum of the real parts of the complex numbers satisfying the equations $\left|\frac{z-4i}{z-2i}\right| = 1$ and $\left|\frac{z-8+3i}{z+3i}\right| = \frac{3}{5}$ is $\frac{k}{5}$, find k.

54. If the equation $z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$ where a_1, a_2, a_3, a_4 are real coefficients different from zero has purely imaginary roots then find the value of the expression $\frac{a_3}{a_1a_2} + \frac{a_1a_4}{a_2a_3}$.

55. $\sum_{j=1}^{n-1} \frac{1}{1-e^{\frac{2\pi ij}{n}}} = \frac{n-1}{k}$, find k. $(i = \sqrt{-1})$

56. If $z_1, z_2, z_3 ,...., z_n$ are in G.P with first term as unity such that $z_1 + z_2 + z_3 + ... + z_n = 0$. Now if $z_1, z_2, z_3,....,z_n$ represents the n vertices of a polygon, then the distance between incentre and circumcentre of the polygon is represented by $4k$. Find k.

57. Let λ, z_0 be two complex numbers. $A(z_1)$, $B(z_2)$, $C(z_3)$ be the vertices of a triangle such that $z_1 = z_0 + \lambda$, $z_2 = z_0 + \lambda e^{i\pi/4}, z_3 = z_0 + \lambda e^{i7\pi/11}$ and $\angle ABC = \frac{3k\pi}{22}$ then the value of k is

58. The argument of $(z-a)(\bar{z}-b)$ is equal to that of $\frac{(\sqrt{3}+i)(1+\sqrt{3}i)}{1+i}$, where a, b are real numbers. If locus of z is a circle with centre $\frac{3+i}{2}$, then find $(a + b)$.

59. If $z = \frac{1}{2}(\sqrt{3}-i)$ then the least positive integral value of 'n' such that $(z^{101} + i^{109})^{106} = z^n$ is 'k', then $\frac{2}{5}k =$

SOLUTIONS

1. (b) : $x^n - 1$

$= (x-1)(x-\alpha_1)(x-\alpha_2).....(x-\alpha_{n-1})$

$= x^n - x^{n-1}(1 + \alpha_1 + ... + \alpha_{n-1})$

$$+x^{n-2}\left(\sum_{1\le i<j\le n-1} \alpha_i\alpha_j + \alpha_1 + \alpha_2 ..... + \alpha_{n-1}\right) + ... - 1 = 0$$

$$\Rightarrow \sum_{1\le i<j\le n-1} \alpha_i\alpha_j + \alpha_1 + \alpha_2 ..... + \alpha_n = 0$$

$$\sum_{1\le i<j\le n-1} \alpha_i \alpha_j = 1$$

2. (d) : Given that $z = \cos\theta + i\sin\theta = e^{i\theta}$

$$\therefore \sum_{m=1}^{15} \text{Im}(z^{2m-1}) = \sum_{m=1}^{15} \text{Im}(e^{i\theta})^{2m-1} = \sum_{m=1}^{15} \text{Im}\, e^{i(2m-1)\theta}$$

$= \sin\theta + \sin 3\theta + \sin 5\theta + \ldots\ldots + \sin 29\theta$

$$= \frac{\sin\left(\frac{\theta+29\theta}{2}\right)\sin\left(\frac{15\times 2\theta}{2}\right)}{\sin\left(\frac{2\theta}{2}\right)}$$

$$= \frac{\sin(15\theta)\sin(15\theta)}{\sin\theta} = \frac{1}{4\sin 2°}$$

3. (b) : Let $z = a + ib$

$\Rightarrow \quad a + ib + \sqrt{a^2+b^2} = 2 + 8i$

$\Rightarrow \quad b = 8,\ a + \sqrt{a^2+64} = 2$

$\Rightarrow \ a^2 + 64 = a^2 - 4a + 4$

$\Rightarrow \ -4a = 60 \quad \Rightarrow \quad a = -15$

$\therefore \ |z| = \sqrt{a^2+b^2} = \sqrt{225+64} = \sqrt{289} = 17$

4. (a) : $|3z + 9 - 7i|$

$= |3z + 6 - 3i + 3 - 4i| \le |3(z + 2 - i)| + |3 - 4i|$

$= 3|z+2-i| + \sqrt{3^2+4^2} = 3(5) + 5 = 20$

5. (d) : Let z_1, z_2 be roots of $2z^2 + 2z + \lambda = 0$

$\Rightarrow \ z_1 + z_2 = -1,\ z_1 z_2 = \frac{\lambda}{2}$

When origin, z_1, z_2 forms equilateral triangle,

we have $z_1^2 + z_2^2 = z_1 z_2$

$(z_1 + z_2)^2 = 3z_1 z_2$

$1 = \frac{3\cdot\lambda}{2} \ \Rightarrow \ \lambda = \frac{2}{3}$

6. (c) : $\bar{z} + i\bar{w} \Rightarrow z - iw = 0 \ \Rightarrow z = iw$

Arg $(zw) = \pi \ \Rightarrow \ \arg(z) + \arg(w) = \pi$

$\Rightarrow \ \arg(iw) + \arg w = \pi \ \Rightarrow \ \arg i + 2\arg w = \pi$

$\frac{\pi}{2} + 2\arg w = \pi \ \Rightarrow \ 2\arg w = \frac{\pi}{2}$

$\Rightarrow \ \arg w = \frac{\pi}{4} \Rightarrow \arg(z) = \frac{3\pi}{4}$

7. (d) : $|z_2 + iz_1| = |z_1| + |z_2| \Rightarrow z_2, iz_1, 0$ are collinear.

$\therefore \ \arg(iz_1) = \arg z_2 \Rightarrow \ \arg i + \arg z_1 = \arg z_2$

$\Rightarrow \ \arg z_2 - \arg z_1 = \frac{\pi}{2}$

Now, $z_3 = \frac{z_2 - iz_1}{1-i}$

$\Rightarrow \ (1 - i)z_3 = z_2 - iz_1$

$\Rightarrow \ z_3 - z_2 = i(z_3 - z_1)$

$\frac{z_3 - z_2}{z_3 - z_1} = i \ \Rightarrow \ \arg\left(\frac{z_3-z_2}{z_3-z_1}\right) = \frac{\pi}{2}$ and $|z_3 - z_2| = |z_3 - z_1|$

$\therefore \ AC = BC \ \therefore \ AB^2 = AC^2 + BC^2$

$\Rightarrow \ 25 = 2AC^2 \Rightarrow \ AC = \frac{5}{\sqrt{2}}$

Required area $= \frac{1}{2}\times\frac{5}{\sqrt{2}}\times\frac{5}{\sqrt{2}} = \frac{25}{4}$ sq. units

8. (a) : $A(5, -4)$, $B(-3, 2)$ subtends an angle $\frac{\pi}{4}$ at $C(z)$ on the circle. Hence $\frac{\pi}{2}$ at centre

$OM \perp AB \ \therefore \ AM = \frac{AB}{2} = \frac{\sqrt{64+36}}{2} = \frac{10}{2} = 5$

Radius $= \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$

9. (d) : Let $x = \frac{3-5i}{2} \ \Rightarrow \ 2x = 3 - 5i$

$\Rightarrow \ (2x - 3)^2 = (-5i)^2 \Rightarrow \ 4x^2 - 12x + 9 = 25i^2$

$\Rightarrow \ 4x^2 - 12x + 34 = 0 \ \Rightarrow \ 2x^2 - 6x + 17 = 0$

$$\begin{array}{r|l} & x+4 \\ 2x^2-6x+17 & 2x^3+2x^2-7x+72 \\ & 2x^3-6x^2+17x \\ & (-)\ (+)\ (-) \\ \hline & 8x^2-24x+72 \\ & 8x^2-24x+68 \\ & (-)\ (+)\ (-) \\ \hline & 4 \end{array}$$

10. (b) : Let $x = \text{cis}\,\alpha$, $y = \text{cis}\,\beta$ and $z = \text{cis}\,\gamma$

Clearly $x + y + z = 0, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

$$x^2 + y^2 + z^2 = (x+y+z)^2 - 2xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 0$$

$\Rightarrow \ \text{cis}2\alpha + \text{cis}2\beta + \text{cis}2\gamma = 0$

$\Rightarrow \ \cos2\alpha + \cos2\beta + \cos2\gamma = 0$

$\Rightarrow \ 1 - 2\sin^2\alpha + 1 - 2\sin^2\beta + 1 - 2\sin^2\gamma = 0$

$\Rightarrow \ \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$

11. (a) : We have, $z_1(z_1^2 - 3z_2^2) = 2$

$\Rightarrow \ z_1^2(z_1^4 + 9z_2^4 - 6z_1^2 z_2^2) = 4$

$(z_1^2)^3 + 9z_1^2 z_2^4 - 6z_1^4 z_2^2 = 4$ ...(i)

Also, $z_2^2(3z_1^2 - z_2^2)^2 = 121$

$\Rightarrow \ (z_2^2)^3 + 9z_2^2 z_1^4 - 6z_1^2 z_2^4 = 121$ ...(ii)

(i) + (ii) $\Rightarrow \ (z_1^2 + z_2^2)^3 = 125 \ \Rightarrow \ z_1^2 + z_2^2 = 5$

12. (c) : Let $z = a + ib$, $b \neq 0$, where Im $z = b$

$$z^5 = (a+ib)^5 = a^5 + {}^5C_1 a^4 bi + {}^5C_2 a^3 b^2 i^2 + {}^5C_3 a^2 b^3 i^3 + {}^5C_4 ab^4 i^4 + i^5 b^5$$

Im $z^5 = 5a^4 b - 10a^2 b^3 + b^5$

$$y = \frac{\text{Im}\, z^5}{\text{Im}^5 z} = 5\left(\frac{a}{b}\right)^4 - 10\left(\frac{a}{b}\right)^2 + 1$$

Let $\left(\frac{a}{b}\right)^2 = x$ (say), $x \in R^+$

$y = 5x^2 - 10x + 1 = 5[x^2 - 2x] + 1 = 5[(x-1)^2] - 4$

Hence $y_{min} = -4$

13. (c) : We have

$$\left|\frac{z_1}{2} + \frac{z_2}{3} + \frac{z_3}{4} + \frac{z_4}{5}\right| = \frac{k}{60}|z_1 z_2 z_3 z_4|\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4}\right|$$

Now, $z_1\bar{z}_1 = 2$, $z_2\bar{z}_2 = 3$, $z_3\bar{z}_3 = 4$ and $z_4\bar{z}_4 = 5$

$$\text{So, } k = \frac{60}{|z_1 z_2 z_3 z_4|} = \frac{60}{\sqrt{2}\sqrt{3}\sqrt{4}\sqrt{5}} = \sqrt{30} = |z_4 z_1 z_2|$$

14. (b) : $\left|\frac{1}{z_2} + \frac{1}{z_1}\right| = \left|\frac{1}{z_2} - \frac{1}{z_1}\right| \Rightarrow |z_1 + z_2| = |z_1 - z_2|$

$\Rightarrow z_1\bar{z}_2 + z_2\bar{z}_1 = 0 \Rightarrow \frac{z_1}{z_2}$ is purely imaginary.

$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2} \Rightarrow \angle POQ = \frac{\pi}{2}$

Circumcentre of ΔPOQ is the midpoint of PQ.

15. (c) : Let $A = 1 + 3\alpha + 5\alpha^2 + 7\alpha^3 + ... + 11\alpha^5 + 13\alpha^6$

$\alpha A = \alpha + 3\alpha^2 + 5\alpha^3 + 7\alpha^4 + ... + 11\alpha^6 + 13\alpha^7$

Now, $(1 - \alpha)A = 1 + 2\alpha + 2\alpha^2 + 2\alpha^3 + ... + 2\alpha^6 - 13\alpha^7$

$= -12 + 2[\alpha + \alpha^2 + ... + \alpha^6] = -14 \Rightarrow A = -\frac{14}{1-\alpha}$

16. (d) : $\left|\frac{z_1+z_2}{z_1-z_2}\right| = 1 \Rightarrow \frac{z_1+z_2}{z_1-z_2} = \cos\alpha + i\sin\alpha$

where α is the argument of $\frac{z_1+z_2}{z_1-z_2}$.

Applying componendo and dividendo, we get

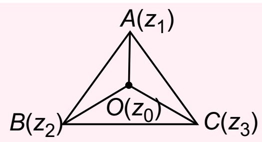
$$\frac{z_1}{z_2} = \frac{1+\cos\alpha + i\sin\alpha}{-1+\cos\alpha + i\sin\alpha}$$

$$= \frac{2\cos\left(\frac{\alpha}{2}\right)\left[\cos\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{\alpha}{2}\right)\right]}{2i\sin\left(\frac{\alpha}{2}\right)\left[\cos\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{\alpha}{2}\right)\right]} = -i\cot\left(\frac{\alpha}{2}\right)$$

Purely imaginary in nature.

17. (c) : Taking rotation at 'O'

$$\frac{z_0 - z_1}{z_0 - z_2} = \cos 2C - i\sin 2C$$



$$\frac{z_0 - z_3}{z_0 - z_2} = \cos 2A + i\sin 2A$$

$$\text{Now } \left(\frac{z_0 - z_1}{z_0 - z_2}\right)\frac{\sin 2A}{\sin 2B} + \left(\frac{z_0 - z_3}{z_0 - z_2}\right)\frac{\sin 2C}{\sin 2B}$$

$$= \frac{\sin 2A\cos 2C - i\sin 2A\sin 2C + \cos 2A\sin 2C + i\sin 2A\sin 2C}{\sin 2B}$$

$$= \frac{\sin(2A+2C)}{\sin 2B} = -1$$

18. (a) : $az^2 + z + 1 = 0$ ... (i)

Taking conjugate of both sides,

$\overline{az^2 + z + 1} = \bar{0} \Rightarrow \bar{a}(\bar{z})^2 + \bar{z} + \bar{1} = 0$

$\bar{a}z^2 - z + 1 = 0$ ...(ii)

(since $\bar{z} = -z$ as 'z' is purely imaginary)

Eliminating 'z' from both the equations, we get

$(\bar{a} - a)^2 + 2(a + \bar{a}) = 0$

Let $a = \cos\theta + i\sin\theta$ (since $|a| = 1$) so that

$(-2i\sin\theta)^2 + 2(2\cos\theta) = 0$

$\Rightarrow \cos\theta = \frac{-1 \pm \sqrt{1+4}}{2}$

Only feasible value of $\cos\theta$ is $\frac{\sqrt{5}-1}{2}$

Hence $a = \cos\theta + i\sin\theta$, where $\theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$

19. (a) : $\cos A = \frac{1}{\sqrt{2}} \Rightarrow A = \frac{\pi}{4}$

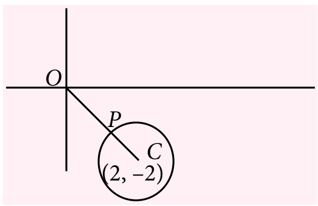
$\therefore \frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| \text{cis}(\pi/4)$

$\Rightarrow \left(\frac{z_3 - z_1}{z_2 - z_1}\right)^4 = \left(\frac{\sqrt{2}}{\sqrt{3}+1}\right)^4 e^{i\pi}$

$\Rightarrow \left(\frac{z_3 - z_1}{z_2 - z_1}\right)^4 = -\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4$

20. (a) : $OP = OC - CP$

$= 2\sqrt{2} - 1$



$\therefore OP : PC = (2\sqrt{2} - 1) : 1$

coordinate of P =

$$\left(\frac{2(2\sqrt{2}-1)}{2\sqrt{2}}, \frac{-2(2\sqrt{2}-1)}{2\sqrt{2}}\right)$$

$$= \left(\left(2 - \frac{1}{\sqrt{2}}\right), -\left(2 - \frac{1}{\sqrt{2}}\right)\right)$$

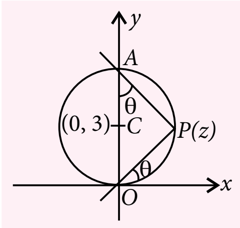
21. (a) : $(z + 1)(z^2 + z + 1) \Rightarrow z = -1, \omega, \omega^2$

Let $f(z) = z^{1985} + z^{100} + 1$

$f(-1) \neq 0$, $f(\omega) = f(\omega^2) = 0 \therefore \omega + \omega^2 = -1$

22. (a) : $r = OA\sin\theta = 6\sin\theta$

$z = 6\sin\theta(\cos\theta + i\sin\theta) \Rightarrow \cot\theta - \frac{6}{z} = i$



23. (d): If $z_1 + \sqrt{z_1^2 - z_2^2} = u$ and $z_1 - \sqrt{z_1^2 - z_2^2} = v$

We have

$$|u|^2 + |v|^2 = \frac{1}{2}|u+v|^2 + \frac{1}{2}|u-v|^2 = 2|z_1|^2 + 2|z_1^2 - z_2^2|$$

And so $(|u| + |v|)^2 = 2\left\{|z_1|^2 + |z_1^2 - z_2^2| + |z_2|^2\right\}$

$$= |z_1 + z_2|^2 + |z_1 - z_2|^2 + 2|z_1^2 - z_2^2|$$

$$= (|z_1 + z_2| + |z_1 - z_2|)^2$$

24. (a) : Let $z_1 = \cos\phi_1 + i\sin\phi_1$ and $z_2 = \cos\phi_2 + i\sin\phi_2$ be the roots of $z^2 + az + b = 0$

$z_1 + z_2 = (-a)$ and $z_1 z_2 = b$

$$-2\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\left[\cos\left(\frac{\phi_1 + \phi_2}{2}\right) + i\sin\left(\frac{\phi_1 + \phi_2}{2}\right)\right] = a$$

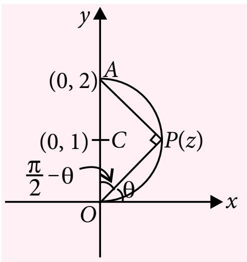
$\Rightarrow \arg a = \frac{\phi_1 + \phi_2}{2}$

$\arg b = \phi_1 + \phi_2$

$\therefore \arg b = 2\arg a$

Also $|z_1 z_2| = |b| = 1$ and $|a| \le 2$

25. (c) : $\angle AOP = \frac{\pi}{2} - \theta$



$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta = \frac{|z|}{2}$

Also $\frac{2i}{z} = \frac{OA}{OP}(\sin\theta + i\cos\theta)$

$= \frac{2}{|z|}(\sin\theta + i\cos\theta)$

$= 1 + i\cot\theta \quad \Rightarrow \frac{2}{z} = -i + \cot\theta \Rightarrow \cot\theta - \frac{2}{z} = i$

26. (a, b, c) : $|z_1| = |z_2| = 1$

$\Rightarrow a^2 + b^2 = c^2 + d^2 = 1$ ...(i)

And $\text{Re}(z_1\bar{z}_2) = 0 \Rightarrow \text{Re}\{(a+ib)(c-id)\} = 0$

$\Rightarrow ac + bd = 0$ ...(ii)

Now from (i) and (ii), we get

$a^2 + b^2 = 1 \Rightarrow a^2 + \frac{a^2c^2}{d^2} = 1 \Rightarrow a^2 = d^2$ ...(iii)

Also $c^2 + d^2 = 1 \Rightarrow c^2 + \frac{a^2c^2}{b^2} = 1 \Rightarrow b^2 = c^2$ ...(iv)

$|\omega_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1$ [From (i) and (iv)]

and $|\omega_2| = \sqrt{b^2 + d^2} = \sqrt{c^2 + d^2} = 1$

[From (i) and (iv)]

Further $\text{Re}(\omega_1\bar{\omega}_2) = \text{Re}\{(a+ic)(b-id)\} = ab + cd = 0$

[From (ii) and (iv)]

Also $\text{Im}(\omega_1\bar{\omega}_2) = bc - ad = \pm 1$

$[\because |\omega_1| = 1, |\omega_2| = 1$ and $\text{Re}(\omega_1\bar{\omega}_2) = 0]$

27. (a, b, c) : $|z_1 + z_2| = |z_1 - z_2|$

$\Rightarrow (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$\Rightarrow z_1\bar{z}_2 + z_2\bar{z}_1 = 0 \Rightarrow \frac{z_1}{z_2} = -\left(\frac{\bar{z}_1}{\bar{z}_2}\right)$

$\Rightarrow \frac{z_1}{z_2}$ is purely imaginary.

Also, $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2$

ΔOAB is a right angled triangle, right angled at O.

So, circumcentre $= \frac{z_1 + z_2}{2}$

28. (a) : $z = (1+\sqrt{2}) + i, \ \bar{z} = (1+\sqrt{2}) - i$

For adjacent vertices

$$z = \bar{z}\,\text{cis}\frac{2\pi}{n} \Rightarrow \text{cis}\frac{2\pi}{n} = \frac{1 + i(\sqrt{2}-1)}{1 - i(\sqrt{2}-1)} = \frac{1 + i\tan\frac{\pi}{8}}{1 - i\tan\frac{\pi}{8}} = \text{cis}\frac{2\pi}{8}$$

$\Rightarrow n = 8$

29. (a, b, d) : Let $|\alpha| = r > 1$ and $\alpha = r\text{cis}\theta$ then

$z = x + iy = \alpha + \frac{1}{\alpha} = r\text{cis}\theta + \frac{\text{cis}(-\theta)}{r}$

$\Rightarrow x = \left(r + \frac{1}{r}\right)\cos\theta$ and $y = \left(r - \frac{1}{r}\right)\sin\theta$

Eliminating θ gives $\frac{x^2}{\left(r + \frac{1}{r}\right)^2} + \frac{y^2}{\left(r - \frac{1}{r}\right)^2} = 1,$

which is an ellipse.

$a = r + \frac{1}{r}, \ b = r - \frac{1}{r} \ (r = |\alpha| > 1 \Rightarrow a > b)$

$\therefore e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{2}{r + \frac{1}{r}}$, distance between foci $= 2ae = 4$

Distance between directrix $= \frac{2a}{e}$

30. (b, d) : $z = 2\cos\frac{3\pi}{5}\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$

$= 2\cos\frac{2\pi}{5}\left(\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right)$

31. (b, c) : $z = (z_1 + z_2 + ... + z_n)\left(\frac{1}{z_1} + \frac{1}{z_2} + ... + \frac{1}{z_n}\right)$

$\Rightarrow\ |z_1 + z_2 + ... + z_n|^2 \rightarrow$ which is real

$\le |z_1|^2 + |z_2|^2 + |z_3|^2 + ... + |z_n|^2 = n^2$

32. (a, d) : $|a| = |b| = |c| = r$

$|c| = |-az^2 - bz| \le r|z|^2 + r|z|$

$\Rightarrow\ |z|^2 + |z| - 1 \ge 0$ ...(i)

and $az^2 = -(bz + c)$

$|a||z|^2 \le r|z| + r$

$\Rightarrow\ |z|^2 - |z| - 1 \le 0$ ...(ii)

Solve (i) and (ii) to conclude that

$\min|z| = \frac{\sqrt{5}-1}{2}$ and $\max|z| = \frac{\sqrt{5}+1}{2}$

33. (a,d) : If $|z - 1| \le |z - 5|$

Then $|z|^2 - z - \bar{z} + 1 \le |z|^2 - 5z - 5\bar{z} + 25\ \&\ |z-3| = |z-1|$

$\Rightarrow\ 4(z+\bar{z}) \le 24$ and $z\bar{z} - 3z - 3\bar{z} + 9 = z\bar{z} - z - \bar{z} + 1$

$\Rightarrow\ z + \bar{z} \le 6$ and $2(z+\bar{z}) = 8 \Rightarrow z + \bar{z} = 4\ \Rightarrow \text{Re}(z) = 2$

$\Rightarrow\ \text{Re}(z) \le 3$

Similarly, for $|z - 1| \ge |z - 5|$, we get $\text{Re}(z) = 4$

34. (a, d) : We observe that if z_0 is a non-zero complex number and c is a real number, then the equation $\bar{z}_0 z + z_0\bar{z} + c = 0$ represents a straight line with complex slope $\frac{-z_0}{\bar{z}_0}$.

Let $L_1 : \bar{\alpha}z + \alpha\bar{z} + c = 0$ and $L_2 : \bar{\beta}z + \beta\bar{z} + d = 0$

where $\alpha = (a, b)$ and $\beta = (p, q)$ are non zero complex numbers. Then their cartesian equations are

$ax + by + \frac{c}{2} = 0$ and $px + qy + \frac{d}{2} = 0$

$\therefore\ L_1 \perp L_2\ \Leftrightarrow\ ap + bq = 0\ \Rightarrow\ \alpha\bar{\beta} + \bar{\alpha}\beta = 0$

$\Leftrightarrow \frac{\alpha}{\bar{\alpha}} + \frac{\beta}{\bar{\beta}} = 0 \Leftrightarrow \mu_1 + \mu_2 = 0$,

where $\mu_1 = -\frac{\alpha}{\bar{\alpha}}$ and $\mu_2 = \frac{-\beta}{\bar{\beta}}$ are the complex slopes of L_1 and L_2 respectively.

$L_1 \| L_2 \Leftrightarrow aq - bp = 0 \Leftrightarrow \alpha\bar{\beta} - \bar{\alpha}\beta = 0$

$\Leftrightarrow \frac{\alpha}{\bar{\alpha}} = \frac{\beta}{\bar{\beta}} \Leftrightarrow \mu_1 = \mu_2$

35. (a) : $z_1 = \cos\theta + i\sin\theta$

$z_2 = \cos 2\theta + i\sin 2\theta - (\cos\theta + i\sin\theta)$

$= (\cos 2\theta - \cos\theta) + i(\sin 2\theta - \sin\theta)$

$|z_2|^2 = (\cos 2\theta - \cos\theta)^2 + (\sin 2\theta - \sin\theta)^2$

$= 2 - 2(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta) = 2 - 2\cos\theta = 4\sin^2\frac{\theta}{2}$

36. (c) : $z_2 = (\cos 2\theta - \cos\theta) + i(\sin 2\theta - \sin\theta)$

$= -2\sin\frac{3\theta}{2}\sin\frac{\theta}{2} + i\,2\cos\frac{3\theta}{2}\sin\frac{\theta}{2}$

$= 2i\sin\frac{\theta}{2}\left(\cos\frac{3\theta}{2} + i\sin\frac{3\theta}{2}\right)$

$\arg z_2 = \arg\left(2i\sin\frac{\theta}{2}\right) + \arg\left(\cos\frac{3\theta}{2} + i\sin\frac{3\theta}{2}\right)$

$= \frac{\pi}{2} + \frac{3\theta}{2}$

$4n\pi < \theta < (4n+2)\pi \Rightarrow 2n\pi < \frac{\theta}{2} < (2n+1)\pi$

$\Rightarrow\ \sin\frac{\theta}{2} > 0$

37. (b) : $(4n + 2)\pi < \theta < (4n + 4)\pi$

$\Rightarrow\ (2n+1)\pi < \frac{\theta}{2} < (2n+2)\pi\ \Rightarrow\ \sin\frac{\theta}{2} < 0$

$\therefore\ \arg z_2 = \frac{3\pi}{2} + \frac{3\theta}{2}$

38. (b) : $\left(2 - 2\cos\frac{2\pi}{n}\right) + \left(2 - 2\cos\frac{4\pi}{n}\right) + ... + \left(2 - 2\cos(n-1)\frac{2\pi}{n}\right)$

$= 2(n-1) - 2\left(\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + ... + \cos\frac{(n-1)2\pi}{n}\right)$

$= 2(n - 1) - 2(-1) = 2n$

39. (b) : Consider the triangle OA_1A_j

Let angle $A_1OA_j = \alpha$

Then $(A_1A_j)^2 = 1^2 + 1^2 - 2(1)(1)\cos\alpha$

$= 2(1 - \cos\alpha) = 2\cdot 2\sin^2\frac{\alpha}{2}$

$\Rightarrow\ A_1A_j = 2\sin\frac{\alpha}{2}$ but $a = \frac{2\pi}{n}(j-1)$

$\Rightarrow\ A_1A_j = 2\sin\frac{(j-1)\pi}{n}$

40. (b) : $2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}.......\sin\frac{(n-1)\pi}{n} = n$

41. (b) : $w = \frac{z-i}{2z+1} = \frac{(2x^2 + x + 2y^2 - 2y) + i(y - 2x - 1)}{(2x+1)^2 + 4y^2}$

$\text{Re}(w) = 0\ \Rightarrow\ x^2 + y^2 + \frac{x}{2} - y = 0$

42. (c) : $\text{Im}(w) = 0\ \Rightarrow\ y = 2x + 1$

43. (c) : $|w| = 1\ \Rightarrow\ w\bar{w} = 1\ \Rightarrow\ x^2 + y^2 + \frac{4x}{3} + \frac{2y}{3} = 0$

44. (a) : Locus of 'z' is perpendicular bisector of the line segment joining (0, 5) and (0, –5). Its equation is $y = 0$ (*i.e.*, x-axis)

45. (b) : Let $w = -i + \frac{15}{z}\ \Rightarrow\ w + i = \frac{15}{z}$

$\Rightarrow\ |w + i| = \frac{15}{|z|} = 3$

Locus of 'w' is a circle with centre (0, –1) and radius 3.

46. (c) : $|z - i| \le 2$ represents a disc with centre at (0, 1) and radius 2.

$|iz + z_1| = |i|\,|z - iz_1| = |z - i(3 + 4i)|$

$= |(z-i)+(4-2i)| \le |z-i| + |4-2i| \le 2+\sqrt{20}$

Maximum value of $|iz + z_1| = \sqrt{20}+2$

47. A → r; B → p; C → s; D → q

(A) $1 + z^3 = 1 + \cos 3\alpha + i\sin 3\alpha$

$= 2\cos\left(\frac{3\alpha}{2}\right)\left[\cos\frac{3\alpha}{2} + i\sin\frac{3\alpha}{2}\right]$

$\Rightarrow \arg(1+z^3) = \frac{3\alpha}{2}$

(B) $1 - z^4 = 1 - \cos 4\alpha - i\sin 4\alpha$

$= 2\sin 2\alpha\left[\cos\left(2\alpha - \frac{\pi}{2}\right) + i\sin\left(2\alpha - \frac{\pi}{2}\right)\right]$

$\Rightarrow \arg(1-z^4) = 2\alpha - \frac{\pi}{2}$

(C) $\frac{1+z^3}{1-z^4} = \frac{\cos\frac{3\alpha}{2}}{\sin 2\alpha}\left[\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right]$

$\Rightarrow \arg\left(\frac{1+z^3}{1-z^4}\right) = \frac{\pi}{2} - \frac{\alpha}{2}$

(D) $\frac{z^4-1}{z^3+1} = \frac{\sin 2\alpha}{\cos\left(\frac{3\alpha}{2}\right)}\left[\cos\left(\frac{\alpha}{2}+\frac{\pi}{2}\right) + i\sin\left(\frac{\alpha}{2}+\frac{\pi}{2}\right)\right]$

$\arg\left(\frac{z^4-1}{z^3+1}\right) = \frac{\pi}{2} + \frac{\alpha}{2}$

48. A → r; B → r; C → q; D → q

Let α be an interior angle of the polygon

$z_n - z_1 = (z_2 - z_1)e^{i\alpha} \Rightarrow z_n = (1-e^{i\alpha})z_1 + e^{i\alpha}z_2$

$z_3 - z_2 = (z_1 - z_2)e^{-i\alpha} \Rightarrow z_3 = e^{-i\alpha}z_1 + (1-e^{-i\alpha})z_2$

$z_3 + z_n = (1 - 2i\sin\alpha)\, z_1 + (1 + 2i\sin\alpha) z_2$

$A = 1 - 2i\sin\alpha$

$|A| = \sqrt{3 - 2\cos 2\alpha} = \sqrt{3 - 2\cos\left(\frac{4\pi}{n}\right)}$

When $n = 4 \Rightarrow |A| = \sqrt{5}$

$n = 6 \Rightarrow |A| = 2$

$n = 8 \Rightarrow |A| = \sqrt{3}$

$n = 12 \Rightarrow |A| = \sqrt{2}$

49. A → s; B → r; C → q; D → q

(A) $|1 - i|^n = 2^n \Rightarrow n/2 = 0 \Rightarrow n = 0$

(B) $x^3 + 2x^2 + 2x + 1 = 0 \Rightarrow x = -1, \omega, \omega^2$

But $x = \omega, \omega^2$ will only satisfy $x^{2000} + x^{2002} + 1 = 0$

(C) $x + 2xy = 0$ and $x^2 - y^2 + y = 0$

$\Rightarrow i, \frac{\sqrt{3}}{2} - \frac{i}{2}, -\frac{\sqrt{3}}{2} - \frac{i}{2}$

(D) $x^2 - y^2 + \sqrt{x^2+y^2} = 0$ and $2xy = 0 \Rightarrow z = 0, i, -i$

50. (5) : Let $\frac{3iz_2}{5z_1} = K$ (real) then $\frac{z_2}{z_1} = \frac{5K}{3i}$

$5\left|\frac{3+7\frac{z_2}{z_1}}{3-7\frac{z_2}{z_1}}\right| = 5\left|\frac{3+\frac{35K}{3i}}{3-\frac{35K}{3i}}\right| = 5\left|\frac{35K+9i}{35K-9i}\right| = 5$

51. (4) : Locus of z is an ellipse

$\frac{(x-11)^2}{25} + \frac{(y-4)^2}{16} = 1$

Equation of tangent is $y - 4 = m(x - 11) + c$

$\Rightarrow c = 11m - 4$

As $c^2 = a^2m^2 + b^2$ for standard ellipse

$\Rightarrow (11m-4)^2 = 25m^2 + 16 \Rightarrow m = 0$ or $m = \frac{11}{12}$

$\therefore \tan\theta = \frac{11}{12} \Rightarrow \theta = \tan^{-1}\frac{11}{12}$

52. (4) : Let α and $i\beta$, $\alpha, \beta \in R$ are roots of $x^2 + ax + b = 0 \Rightarrow \alpha + i\beta = -a,\ i\alpha\beta = b$

$\alpha - i\beta = -\bar{a}$

$\Rightarrow 2\alpha = -(a+\bar{a})$ and $2i\beta = -(a-\bar{a})$

$\therefore 4i\alpha\beta = a^2 - \bar{a}^2 \Rightarrow 4b = a^2 - \bar{a}^2$

53. (5) : $\left|\frac{z-4i}{z-2i}\right| = 1 \Rightarrow z = x + 3i$, using this in

$\left|\frac{z-8+3i}{z+3i}\right| = \frac{3}{5} \Rightarrow 5|x-8+6i| = 3|x+6i|$

$\Rightarrow x = 8, 17$

Two complex numbers are $8 + 3i$, $17 + 3i$

Sum of real parts = 8 + 17 =25

54. (1) : Let $z = iy$

$\Rightarrow y^4 - a_1y^3i - a_2y^2 + ia_3y + a_4 = 0$

$\Rightarrow y^4 - a_2y^2 + a_4 = 0$ ....(i) and $-a_1y^3 + a_3y = 0$

$\Rightarrow y = 0$ or $y^2 = \frac{a_3}{a_1}$ ....(ii)

From (i) and (ii), we get

$\frac{a_3^2}{a_1^2} - \frac{a_2a_3}{a_1} + a_4 = 0 \Rightarrow \frac{a_3}{a_1a_2} + \frac{a_1a_4}{a_2a_3} = 1$

55. (2) : Let $e^{i\frac{2\pi}{n}} = \alpha$ then

$\sum_{j=1}^{n-1}\frac{1}{1-e^{\frac{2\pi}{n}ij}} = \frac{1}{1-\alpha} + \frac{1}{1-\alpha^2} + \ldots\ldots + \frac{1}{1-\alpha^{n-1}}$

where α is a n^{th} root of unity. $(\alpha, \alpha^2, \alpha^3, ..., \alpha^{n-1})$ are the roots of $\frac{x^n-1}{x-1} = (x-\alpha)(x-\alpha^2).......(x-\alpha^{n-1})$

Taking log on both sides

$$\log\frac{x^n-1}{x-1} = \log(x-\alpha) + \log(x-\alpha^2) + ... + \log(x-\alpha^{n-1})$$

Differentiating w.r.t. x and use $\lim\limits_{x\to 1}$

$$\Rightarrow \quad \frac{n-1}{2} = \frac{1}{1-\alpha} + \frac{1}{1-\alpha^2} + ... + \frac{1}{1-\alpha^{n-1}}$$

56. (0) : Let vertices be $1, \alpha, \alpha^2, ..., \alpha^{n-1}$

Given $1 + \alpha + \alpha^2 + ... \alpha^{n-1} = 0 \Rightarrow \alpha^n - 1 = 0$

$\Rightarrow z_1, z_2, z_3, ..., z_n$ are roots of $\alpha^n = 1$

which form regular polygon. So, distance is zero.

57. (5) : $|z_1 - z_0| = |z_2 - z_0| = |z_3 - z_0| = |\lambda|$

$$\frac{z_3 - z_0}{z_2 - z_0} = \frac{e^{i7\pi/11}}{e^{i\pi/4}} = e^{i17\pi/44}$$

$$\Rightarrow \quad \angle BSC = \frac{17\pi}{44} \Rightarrow \angle BAC = \frac{17\pi}{88}$$

Similarly $\frac{z_2 - z_0}{z_1 - z_0} = e^{i\pi/4} \Rightarrow \angle ACB = \frac{\pi}{8}$

$$\therefore \quad \angle ABC = \pi - \frac{\pi}{8} - \frac{17\pi}{88} = \frac{15\pi}{22}$$

58. (3) : $\tan^{-1}\frac{(a-b)y}{x^2 + y^2 - (a+b)x + ab} = \frac{\pi}{4}$

$\Rightarrow x^2 + y^2 - (a + b)x - (a - b)y + ab = 0$

Centre $= \frac{3+i}{2} \Rightarrow a + b = 3$

59. (4) : $z = \frac{-1}{2}i(1 + i\sqrt{3}) = i\omega^2$

$z^{101} = i\omega$

$(z^{101} + i^{109})^{106} = (-i\omega^2)^{106} = -\omega^2$

$-\omega^2 = (i\omega^2)^n = i^n\omega^{2n}$ or $\omega^{2n-2}\, i^n = -1$

This is possible only when $n = 4r + 2$ and $2n - 2$ is multiple of 3 *i.e.*,

$2(4r + 2) - 2$ is a multiple of 3.

i.e., $8r + 2$ is a multiple of $3 \Rightarrow r = 2$

$\therefore \quad n = 10$

$\therefore \quad \frac{2}{5}k = 4$